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TESTING EXPONENTIALITY AGAINST HNBUET CLASS BASED ON GOODNESS OF FITAPPROACH

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Abstract

Based on goodness of fit approach a new test for testing exponentiality against harmonic new better than used in expectation upper tail (HNBUET) is proposed. The percentiles of this test are calculated and tabulated for sample size $n-5(1)50$. For some commonly used life distributions in reliability such as linear failure rate, Makeham and Weibull distributions, the Pitman asymptotic efficiency (*PAE*) and the power of proposed test are calculated. Finally the proposed test is applied to two sets of real data.

Keywords: HNBUET;Classes of life distributions; test exponentiality; Goodness of fit; Pitman's asymptotic efficiency.

1 Introduction

Ever since the work of Barlow et al. (1963) and Bryson and siddiqui (1969), various classes of life distributions have been introduced in reliability. These classes of life distributions can be applied in engineering, maintenance and medicine. A growing interest in modelling survival data using classification of the life distributions based on some aspects of aging have been made by some statistics and reliability analysts. Of the most common and practical aspects are IFR (increasing failure rate), IFRA (increasing failure rate average), NBU (new better than used), NBUE (new better than used in expectation) andHNBUE (harmonic new better than used in expectation). For definitions of these classes and further details see, e.g., Barlow and Proschan (1981) and Zacks (1992).

A new class of life distribution named HNBUE (3) (harmonic new better than used in expectation of third order) which is larger than HNBUE class has been introduced by Deshpande et al. (1986). This class has been renamed asHNBUET (harmonic new better than used in expectation upper tail) by Abouammoh and Ahmed (1989).

$$\text{IFR} \Longrightarrow \text{IFRA} \Longrightarrow \text{NBU} \Longrightarrow \text{NBUE} \Longrightarrow \text{HNBUE} \Longrightarrow \text{HNBUET}$$

Testing exponentiality based on goodness of fit technique versus many classes of life distributions was taken up by some authors such as Ahmed and Alwasel (1999), Ahmad et al. (2001), El- Bassiouny and Alwasel (2003),Hendi and Al-

Ghufily (2004), Ismail and Abu- Youssef (2012) , Mahmoud and Abdul Alim (2006), Abu- Youssef (2009), Mahmoud and Diab(2008),Diab (2010).

2 TestingExponentiality Against HNBUET

In this section the following hypotheses will be tested

H_0 : F is exponential.

and

H_1 : F belongs to HNBUET and not exponential.

Recall that *F* is HNBUET iff

$$\int_x^\infty \upsilon(y)dy \le \mu^2 e^{-x/\mu}, \qquad x > 0, \quad \mu > 0.$$

Where $\upsilon(y) = \int_y^\infty \bar{F}(u)\,du$.For more details see Abouammoh and Ahmed (1989).

We need to state and prove the following theorem.

Theorem 2.1. Let X be HNBUET random variable with distribution *F*, then

$$\frac{1}{2}(\mu+1)\mu_2 - \mu^2 - E(e^{-X})(\mu+1) + 1 \le \mu^3, \tag{1}$$

Where μ and $\mu_2$are the first and the second moments of the distribution *F*.

Proof.

Consider the following integral

$$\int_0^\infty \int_x^\infty \upsilon(y)dy\, e^{-x}\, dx \le \mu^2 \int_0^\infty e^{-x(\mu+1)/\mu}\, dx\ . \tag{2}$$

Setting

$$\mathrm{I} = \int_0^\infty \int_x^\infty \upsilon(y)dy\, e^{-x}\, dx,$$

and

$$\mathrm{II} = \mu^2 \int_0^\infty e^{-x(\mu+1)/\mu}\, dx.$$

The integral I can be put in the following form

$$\mathrm{I} = \int_0^\infty v(x) \int_0^x e^{-y}\, dy\, dx,$$

therefore

$$\mathrm{I} = \int_0^\infty x\, \bar{F}(x) dx - \int_0^\infty \bar{F}(x) dx + \int_0^\infty \bar{F}(x) e^{-x}\, dx, \tag{3}$$

Then,

$$\mathrm{I} = \frac{1}{2}\mu_2 - \mu + 1 - E(e^{-X}). \tag{4}$$

The integral II can be put in the following form

$$\mathrm{II} = \frac{\mu^3}{\mu+1} \int_0^\infty e^{-y}\, dy,$$

there fore

$$\mathrm{II} = \frac{\mu^3}{\mu+1}. \tag{5}$$

substituting (4) and (5) into (2), we get

$$\frac{1}{2}(\mu+1)\mu_2 - \mu^2 + 1 - E(e^{-X})(\mu+1) \le \mu^3.$$

This completes the proof.

2.1 Empirical test statistic for HNBUET

Let $X_1, X_2, \dots, X_n$ be a random sample from $F \in$ HNBUET class and $\hat{\delta}^{(1)}_{HNB}$ be the empirical estimate of $\delta^{(1)}_{HNB}$,where

$$\delta^{(1)}_{HNB} = \mu^3 - \frac{1}{2}(\mu+1)\mu_2 + \mu^2 - 1 + \quad E(e^{-X})(\mu+1)$$

It is easy to show that

$$\hat{\delta}^{(1)}_{HNB} = \frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[X_i X_j X_k - \frac{1}{2}(X_i+1)X_j^2 + X_i X_j + (X_i+1)e^{-X_j} - 1\right].$$

Setting

$$\emptyset(X_1, X_2, X_3) = X_1 X_2 X_3 - \frac{1}{2}(X_1+1)X_2^2 + X_1 X_2 - (X_1+1)e^{-X_2} - 1,$$

and defining symmetrical kernel

$$\psi(X_1,X_2,X_3)=\frac{1}{3!}\sum \emptyset(X_{i1},X_{i2},X_{i3}),$$

where the summation is over all arrangements of X_{i1},X_{i2},X_{i3} , then $\delta_{HNB}^{(1)}$ is equivalent to U-statistic

$$U_n^{(3)}=\frac{1}{\binom{n}{3}}\sum_{i\neq j\neq k}^{n}\psi(X_1,X_2,X_3).$$

The following theorem summarize the asymptotic properties of the test

Theorem 2.2

(i) As $n\to\infty$, $\sqrt{n}\,(\hat{\delta}_{HNB}^{(1)}-\delta_{HNB}^{(1)})$ is asymptotically normal with mean 0 and variance

$$\sigma^2=var\left\{3X\mu^2+2X\mu-\frac{1}{2}X\mu_2-\mu_2+\mu^2-\frac{1}{2}X^2\mu-\frac{1}{2}X^2-\frac{1}{2}\mu\mu_2+\mu e^{-X}+e^{-X}\right.$$
$$\left.+X\,E(e^{-X})+2E(e^{-X})+\mu E(e^{-X})-3\right\}. \qquad (6)$$

(ii) Under H_0 , the variance is

$$\sigma_0^2=16.583$$

Proof.

(i)Using standard U-statistics theory, see Lee (1990), and by direct calculations we can find the mean and the variance as follows

$$\sigma^2=var\{\eta(X)\}, \qquad (7)$$

Where

$$\eta(X)=\eta_1(X)+\eta_2(X)+\eta_3(X),$$
$$\eta_1(X)=E\big(\emptyset(X_1,X_2,X_3)\big|X_1\big),$$
$$\eta_2(X)=E\big(\emptyset(X_1,X_2,X_3)\big|X_2\big),$$

and $\eta_3(X)=E\big(\emptyset(X_1,X_2,X_3)\big|X_3\big)$

thus

$$\eta_1(X)=X\mu^2-\frac{1}{2}(X+1)\mu_2+X\mu+E(e^{-X})(X+1)-1, \qquad (8)$$

$$\eta_2(X)=X\mu^2-\frac{1}{2}(\mu+1)X^2+X\mu+e^{-X}(\mu+1)-1, \qquad (9)$$

and

$$\eta_3(X)=X\mu^2-\frac{1}{2}(\mu+1)\mu_2+\mu^2+E(e^{-X})(\mu+1)-1. \qquad (10)$$

Upon using (7), (8), (9) and (10), Eq. (6) is obtained.

(ii) UnderH_0, it is easy to prove that $E\left(\eta(X)\right)=0$ and$\sigma_0^2=E\left(\left(\eta(X)\right)^2\right)$.

After some calculations we obtain

$\sigma_0=4.072$.

3 The Pitman Asymptotic efficiency

In this section, we evaluate the Pitman asymptotic efficiency (*PAE*) of the test $\delta_{HNB}^{(1)}$ for three alternatives to assess the qualityof this test. These are linear failure rate, Makeham and weibull alternatives. We choose them since they are in the HNBUET class. We have

$$\delta_{HNB}^{(1)}=\mu^3-\frac{1}{2}(\mu+1)\mu_2+\mu^2+E(e^{-X})(\mu+1)-1.$$

To evaluate the *PAE* consider

$$\delta_{HNB}^{(1)}(\theta)=\mu_\theta^3-\frac{1}{2}(\mu_\theta+1)\mu_{2\theta}+\mu_\theta^2+E_\theta(e^{-X})(\mu_\theta+1)-1, \qquad (11)$$

where

$$\mu_\theta=\int_0^\infty \bar{F}_\theta(x)dx,$$

$$E_\theta(e^{-x})=\int_0^\infty e^{-x}dF_\theta(x),$$

and

$$\mu_{2\theta}=2\int_0^\infty x\,\bar{F}_\theta(x)dx.$$

(i) Linear failure rate family:

$$\bar{F}_1(x)=exp\left(-x-\frac{\theta x^2}{2}\right), \qquad x>0,\theta\geq 0$$

(ii) Makeham family:

$$\bar{F}_2(x)=exp[-x+\theta(x+e^{-x}-1)], \qquad x>0,\theta\geq 0$$

(iii) Wiebull family:

$$\bar{F}_3(x)=exp\left(-x^\theta\right), \qquad x>0,\theta\geq 1$$

Let *T* be a test statistic for testing H_0: $F_0\in\{F_{n_\theta}\}$, $n_\theta=\theta+cn^{-1/n}$, where c is an arbitrary const, then the Pitman asymptotic efficiency of *T* is as follows

$$PAE(T) = \frac{\acute{T}(\theta)}{\sigma_{\circ}(\theta)}\Bigg|_{\theta=\theta_{\circ}}, \qquad where \quad \acute{-} = \frac{d}{d\theta} \tag{12}$$

Differentiating both sides of (11) w.r.t θ gives

$$\frac{d}{d\theta}\delta_{HNB}^{(1)}(\theta) = 3\mu_\theta^2\acute{\mu}_\theta - \frac{1}{2}\mu_\theta\acute{\mu}_{2\theta} - \frac{1}{2}\mu_{2\theta}\acute{\mu}_\theta - \frac{1}{2}\acute{\mu}_{2\theta} + 2\mu_\theta\acute{\mu}_\theta + \acute{\mu}_\theta\int_0^\infty e^{-x}\,dF_\theta(x)$$
$$+\mu_\theta\int_0^\infty e^{-x}\,d\acute{F}_\theta(x) + \int_0^\infty e^{-x}\,d\acute{F}_\theta(x), \tag{13}$$

3.1 $PAE\left(\delta_{HNB}^{(1)}\right)$ for LFR family

Using (12) and (13) considering $\theta_{\circ} = 0$, we obtain

$$PAE\left(\delta_{HNB}^{(1)}\right) = 0.4298$$

3.2 $PAE\left(\delta_{HNB}^{(1)}\right)$ for Makeham family

Using (12) and (13) considering $\theta_{\circ} = 0$, we obtain

$PAE\left(\delta_{HNB}^{(1)}\right) = 0.143$

3.3 $PAE\left(\delta_{HNB}^{(1)}\right)$ for Wiebull family

Using (12) and (13) considering $\theta_{\circ} = 1$, we obtain

$PAE\left(\delta_{HNB}^{(1)}\right) = 0.472$

4 Monte Carlo Null Distribution Critical Points

In this section the upper percentile points of $\delta_{HNB}^{(1)}$ for 90%, 95%, 98% and 99% are calculated based on 10000 simulated samples of sizes n=5(1)50 and tabulated in Table 1 .

Table 1: Critical Values of the Statistic $\delta_{HNB}^{(1)}$

n	0.90	0.95	0.98	0.99
5	1.2955	2.0126	3.2559	4.2802
6	1.1139	1.7304	2.6378	3.4173
7	1.0135	1.5305	2.3637	3.0208
8	0.9227	1.3778	1.9897	2.5297
9	0.8076	1.1588	1.6917	2.1960
10	0.7990	1.1215	1.5938	2.0702
11	0.7338	1.0266	1.4586	1.8359
12	0.6760	0.9549	1.3220	1.6608
13	0.6277	0.8613	1.1793	1.4489
14	0.6027	0.8349	1.1966	1.4765
15	0.5827	0.8072	1.1254	1.3708
16	0.5837	0.7827	1.0859	1.3226
17	0.5433	0.7310	1.0121	1.2217
18	0.5172	0.6973	0.9598	1.1558
19	0.4878	0.6563	0.9182	1.1172
20	0.4899	0.6524	0.8933	1.0649
21	0.4698	0.6255	0.8468	1.0195
22	0.4711	0.6075	0.8298	1.0095
23	0.4562	0.6056	0.8241	0.9949
24	0.4410	0.5743	0.7896	0.9271
25	0.4305	0.5556	0.7316	0.8914
26	0.4241	0.5522	0.7311	0.8758
27	0.4104	0.5340	0.6973	0.8206
28	0.4069	0.5251	0.6865	0.8109
29	0.3902	0.5062	0.6703	0.7904
30	0.3899	0.5125	0.6608	0.7877
31	0.3863	0.4903	0.6522	0.7494
32	0.3749	0.4869	0.6451	0.7696
33	0.3662	0.4705	0.6185	0.7140
34	0.3594	0.4547	0.6042	0.7021
35	0.3446	0.4480	0.5803	0.6878
36	0.3501	0.4539	0.5773	0.6664
37	0.3483	0.4420	0.5872	0.6779
38	0.3406	0.4280	0.5514	0.6429
39	0.3423	0.4328	0.5572	0.6431
40	0.3319	0.4187	0.5417	0.6448
41	0.3291	0.4210	0.5502	0.6316
42	0.3182	0.4083	0.5310	0.6176
43	0.3213	0.4110	0.5257	0.6013
44	0.3161	0.4010	0.5219	0.6114
45	0.3122	0.4012	0.5136	0.6011
46	0.3081	0.3928	0.5075	0.5811
47	0.3019	0.3850	0.4834	0.5565
48	0.2962	0.3815	0.4893	0.5558
49	0.3026	0.3807	0.4721	0.5405
50	0.2982	0.3792	0.4714	0.5489

5 The power Estimates of $\delta_{HNB}^{(1)}$

The power of the statistic $\delta_{HNB}^{(1)}$ is considered at the significant level $\alpha = 0.05$ for three commonly used distributions such as linear failure rate, Makeham and Weibull distributions. These estimates are based on *10000* simulated samples of size *n=10,20*and *30* and tabulated in Table 2

Table 2: Power Estimates of the Statistic $\delta_{HNB}^{(1)}$

N	θ	LFR	Makeham	Weibull
10	1	1.0000	1.0000	1.0000
	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
20	1	1.0000	1.0000	1.0000
	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
30	1	1.0000	1.0000	1.0000
	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000

From Table 2, we see that our test $\delta_{HNB}^{(1)}$ has a good power for all alternatives.

6 Pitman Asymptotic Relative Efficiency (PARE)

In this section we compare our test statistic $\delta_{HNB}^{(2)}$, by using Pitman asymptotic relative efficiency for linear (LFR), Makeham and wiebull families alternatives with the tests δ_3 and $\delta_{Fn}^{(2)}$ given by Mugdadi and Ahmad (2005) Mahmoud and Abdul Alim (2008) respectively.

Table 3: Pitman Asymptotic Relative Efficiency

distribution	δ_3	$\delta_{Fn}^{(2)}$	$\delta_{HNB}^{(1)}$	PARE $(\delta_{HNB}^{(1)}, \delta_3)$	PARE $(\delta_{HNB}^{(1)}, \delta_{Fn}^{(2)})$,
F_1:LFR	0.408	0.217	0.4298	1.053	1.981
F_2:Makeham	0.0395	0.144	0.143	3.620	0.993
F_3:Weibull	0.170	0.05	0.472	2.777	9.44

Table 3 shows that the proposed test performs well comparing with the tests δ_3 and $\delta_{Fn}^{(2)}$ for all alternatives.

7 Applications

In all applications we test the null hypothesis that the life distribution is exponential versus the alternative that the life distribution is HNBUET and not exponential. Also we use $\alpha = 0.05$ in all examples

Example 1

Consider the data in Abouammoh et al.(1994), these data represent set of 40 patients suffering from blood cancer (Leukemia) from one ministry of health hospital in Saudi Arabia and order values in years are:

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162
2.211	2.370	2.532	2.693	2.805	2.910	2.912	3.192	3.263	3.348
3.348	3.427	3.499	3.534	3.767	3.751	3.858	3.986	4.049	4.244
4.323	4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074	4.381

It was found that

$\delta_{HNB}^{(1)} = 15.9033$which exceeds the critical value in Table 1.

Then we reject the null hypothesis of exponentiality and accept H_1 which states that the data have HNBUET property.

Example 2

The following data was considered by Pavur et al. (1992). The results recorded in the following table are the number of revolutions (in ten millions) to failure of 23 ball bearings in a life test study.

1.788	2.892	3.300	4.152	4.212	4.560	4.848	5.184	5.196	5.412
5.556	6.78	6.864	6.864	6.988	8.412	9.312	9.864	10.512	10.584
12.792	12.804	17.340							

It was found that

$\delta_{HNB}^{(1)} = 158.6605$which exceeds the critical value in Table 1.

Then we reject the null hypothesis of exponentiality and acceptH_1 which states that the data have HNBUET property.

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