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Notes on Admissible Solutions of a Class of Differential Equations

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Abstract

Aims/ objectives: Let Q_1, Q_2 and Q_3 be three non-vanishing meromorphic functions, and α be an entire function. In this paper, we consider the differential equation of the form:

$$f^{(k)}(z) - Q_1(z) = (f(z) - Q_2(z))Q_3(z)e^{\alpha(z)},$$

and derive some necessary conditions (in terms of Q_j $(j = 1, 2, 3)$ and α) for the existence of an admissible meromorphic solution f of the above equation. As a consequence of the studies, particularly we are able to confirm partially the validity of Brück Conjecture raised in studying value sharing of an entire function and its first derivative.

Keywords: Nevanlinna theory; admissible solutions; differential equation; sharing value; brück conjecture.

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1 Introduction and Main Results

Let f denote a nonconstant meromorphic function. We assume that the reader is familiar with the basic Nevanlinna's value distribution theory of meromorphic functions and its standard notations.

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Such as, the characteristic function

$$T(r,f) = N(r,f) + m(r,f),$$

the counting function of the poles

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r,$$

the proximity function

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where

$$\log^+ x = \log\max\{x,1\} = \max\{\log x, 0\}\ (x \geq 0),$$

and etc., see, e.g., [1] and [2].

A meromorphic function β is called recall a small function of f, if $T(r,\beta) = S(r,f)$, where $S(r,f)$ denotes any quantity that satisfies $S(r,f) = o(1)T(r,f)$ as $r \to \infty$, possibly outside a set of r of finite linear measure.

In addition, let f and g be two nonconstant meromorphic functions, $a \in \mathbb{C} \cup \{\infty\}$. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, we say that they share the value or some function a CM (Counting multiplicities). We also need the following concepts.

Definition 1.1 The order $\rho(f)$ and the hyper-order $\rho_2(f)$ of a meromorphic function f are defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

respectively.

The subject on sharing values between entire functions and their derivatives was first studied by Rubel-Yang [3]. They proved a result in 1977 that if a non-constant entire function f and f' share two distinct finite numbers a, b CM, then $f = f'$. Since then, shared value problems have been studied by many authors and a number of profound results have been obtained, see, e.g., [2].

Later on, Brück [4] studied the relationships between f and f' when they share only one finite value CM. Meanwhile, the following conjecture was posed in [4].

Brück Conjecture. Let f be a non-constant entire function such that the hyper-order $\rho_2(f)$ of f is finite and non-integer. If f and f' share a finite value $a\,(\neq 0)$ CM, then

$$f' - a = c(f - a) \tag{1.1}$$

or

$$f' - cf = a - ca, \tag{1.2}$$

for some constant $c \neq 0$.

If $a = 0$, the above conjecture was proved by Brück [4], and he also proved the validity of the conjecture when $a \neq 0$, provided that f satisfies the following additional condition:

$$N(r, 1/f') = S(r,f). \tag{1.3}$$

Moreover, for the case that f is of finite order, the conjecture had been proved by Gundersen-Yang [5]. Later on, Chen-Shon [6] confirmed the conjecture when the order of f $\rho(f) = \infty$ and $\rho_2(f) < 1/2$. Most recently, without any restriction on the order or growth of f, Li-Gao-Zhang [7] proved the following result:

Theorem A. Let f be a non-constant entire function. If f and $f^{(k)}$ $(k \geq 1)$ share the value 1 CM, and if

$$N(r, \frac{1}{f^{(k)}}) < \alpha T(r, f) \text{ (for } r \geq r_0), \tag{1.4}$$

where $\alpha \in [0, 1/4)$, then

$$f^{(k)} - 1 = c(f - 1), \tag{1.5}$$

for some nonzero constant c.

Note that both conditions (1.3) and (1.4) indicate, for $k \geq 1$,

$$N(r, \frac{1}{f^{(k)}}) < T(r, f) \text{ (for } r \geq r_0), \tag{1.6}$$

which implies, for $k \geq 1$,

$$m(r, \frac{1}{f^{(k)}}) \neq S(r, f). \tag{1.7}$$

Among many interesting applications of the Nevanlinna theory, there are studies on the growth and existence of entire or meromorphic solutions of various types of differential equations. In this paper, we shall tackle for entire functions f, without any restrictions on the order or growth of f that satisfy the following condition:

$$m(r, \frac{1}{f}) \neq S(r, f), \tag{1.8}$$

which is the case ($k = 0$ in (1.7)) that, has been excluded in all the previous studies.

Our main result can be stated as follows:

Theorem 1.1. Consider the following differential equation

$$f^{(k)}(z) - Q_1(z) = (f(z) - Q_2(z))Q_3(z)e^{\alpha(z)}, \tag{1.9}$$

where α is an entire function, Q_j $(j = 1, 2, 3)$ are non-vanishing meromorphic functions. If the equation (1.9) admits an admissible solution f such that

$$m(r, \frac{1}{f}) \neq S(r, f), \tag{1.10}$$

then the following condition must be satisfied:

$$Q_3 e^{\alpha} = \frac{Q_1}{Q_2}. \tag{1.11}$$

Furthermore, equation (1.9) can then be reduced to the following simplified form:

$$f^{(k)} = \frac{Q_1}{Q_2} f. \tag{1.12}$$

In 2004, Liu-Gu [8] improved the result of Yu [9] and proved the following theorem:

Theorem B. Let $k \geq 1$ and let f be a non-constant entire function, a be a meromorphic function with $a \not\equiv 0$, ∞, and $T(r, a) = S(r, f)$ as $r \to \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0, f) > 1/2$, then $f \equiv f^{(k)}$.

As a consequence of the above theorem, we have the following result which is an improvement over Theorem B.

Corollary 1.1. Let f be an entire function with $m(r, 1/f) \neq S(r, f)$, $a\,(a \not\equiv 0, \infty)$ be a small function of f. If $f - a$ and $f^{(k)} - a$ (for some $k \geq 1$) share the value 0 CM, then $f \equiv f^{(k)}$.

2 Lemma and Proof of Theorem 1.1

The following lemma is crucial to the proof of our theorem.

Lemma 2.1 [10]. Let f be a meromorphic solution of an algebraic equation

$$P(z, f, f', \cdots, f^{(n)}) = 0, \tag{2.1}$$

where P is a polynomial in $f, f', \cdots, f^{(n)}$ with meromorphic coefficients small with respect to f. If a complex constant c does not satisfy equation (2.1), then

$$m(r, \frac{1}{f - c}) = S(r, f).$$

Proof of Theorem 1.1. Let f be an admissible meromorphic solution of equation (1.9). By taking the logarithmic derivative on both sides of (1.9), we would have

$$\frac{f^{(k+1)} - Q_1'}{f^{(k)} - Q_1} = \frac{Q_3'}{Q_3} + \alpha' + \frac{f' - Q_2'}{f - Q_2},$$

thus

$$\begin{aligned} & f^{(k+1)}f - (\frac{Q_3'}{Q_3} + \alpha')f^{(k)}f - f^{(k)}f' \\ = & \{Q_1' - (\frac{Q_3'}{Q_3} + \alpha')Q_1\}f + Q_2f^{(k+1)} + Q_1f' \\ & -\{(\frac{Q_3'}{Q_3} + \alpha')Q_2 + Q_2'\}f^{(k)} + T, \end{aligned} \tag{2.2}$$

where

$$T := (\frac{Q_3'}{Q_3} + \alpha')Q_1Q_2 + Q_1Q_2' - Q_1'Q_2. \tag{2.3}$$

Obviously, T is a small function of f.

Assume that $T \not\equiv 0$. Then from equation (2.2), the fact that α' is a small function of f, and Lemma 2.1 (where $c = 0$ is used), we would be able to conclude $m(r, \frac{1}{f}) = S(r, f)$, a contradiction to the assumption (1.10). Thus $T \equiv 0$, and hence

$$\frac{Q_3'}{Q_3} + \alpha' = \frac{Q_1'}{Q_1} - \frac{Q_2'}{Q_2}, \tag{2.4}$$

which leads to

$$Q_3e^{\alpha} = A\frac{Q_1}{Q_2}, \tag{2.5}$$

for some nonzero constant A. It follows from this, equation (1.9) becomes

$$f^{(k)} - A\frac{Q_1}{Q_2}f = (1 - A)Q_1. \tag{2.6}$$

Again, by applying Lemma 2.1 to equation (2.6), the constant A must be equal to 1. It follows that

$$Q_3 e^{\alpha} = \frac{Q_1}{Q_2} \quad \text{and} \quad f^{(k)} = \frac{Q_1}{Q_2} f.$$

This also completes the proof of Theorem 1.1.

Remark. The following examples show that the condition (1.10) is necessary for the reduction of equation (1.9) to the form (1.12).
Example 2.1. Let $f(z) = \mathrm{e}^{2z} + z^2$, $Q_1(z) = 2z$, $Q_2(z) = z^2$, and $Q_3(z)\mathrm{e}^{\alpha(z)} \equiv 2$, then it is immediately yields $f'(z) - 2z = 2(f(z) - z^2)$. We find $m(r, 1/f) = S(r, f)$, and $f'(z) \not\equiv (2/z)f(z)$.

Example 2.2. Let $f(z) = \mathrm{e}^{\mathrm{e}^z} + \mathrm{e}^z + 1$, it is easy to see that f is a solution of the following equation

$$f' - \mathrm{e}^z = \mathrm{e}^z(f - \mathrm{e}^z - 1).$$

By the second fundamental theorem for three small functions [1] and [2] , we would get $m(r, 1/f) = S(r, f)$. While, $f'(z) \not\equiv \frac{\mathrm{e}^z}{\mathrm{e}^z+1} f(z)$.

3 Remarks and a Conjecture

Remark 3.1 . We refer the reader to [11] and [12], where complete different arguments are used to study equation (1.9), without the condition (1.10) imposed on f, but f is assumed to be in the form $f = F^n$, for some entire function. In this direction, many results have been obtained by others, see, e.g., [13]-[16]. Now by Theorem 1.1, and using the Tumura-Clunie theorem [17], we can get the following conclusion:

Theorem 3.1. Suppose that F is a transcendental meromorphic function, and that $f = \sum_{j=0}^{n} a_j F^j$, in which a_j $(j = 0, 1, \cdots, n, a_n \neq 0)$ are constants. Assume that f' and f share Q CM, that $m(r, 1/f) \neq S(r, f)$, where $Q \not\equiv 0, \infty$ is a small function of f. Then

$$F = c\,\mathrm{e}^{z/n} - \frac{a_{n-1}}{n a_n},$$

where c is a nonzero constant.

Remark 3.2 . What can be said about the possible relationships between f and $f^{(k)}$, if the condition (1.10) "$m(r, 1/f) \neq S(r, f)$" is replaced by "$m(r, 1/f^{(k)}) \neq S(r, f)$" in Theorem 1.1?

Based on Corollary 1.1, we pose the following more general conjecture, for all the transcendental entire functions, without any restrictions on the order or growth of f.

Conjecture 3.1. Let f be an arbitrary nonconstant entire function. If f and $f^{(k)}$ share the value 1 CM and $m(r, 1/f^{(k)}) \neq S(r, f)$, then $f \equiv f^{(k)}$.

4 Conclusion

In 1996, Brück studied the relation between f and f' if an entire function f shares only one finite value CM with its derivative f'. Meanwhile, he posed a famous conjecture. Since then, under some additional assumptions, many results related to Brück conjecture have been obtained. But unfortunately, at the moment, Brück's conjecture is still unsolved. Maybe much more intricate tools than the standard tools from value distribution theory are required to solve it.

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Competing Interests

The authors declare that no competing interests exist.

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