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One Step Trigonometrically-fitted Third Derivative Method with Oscillatory Solutions

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Authors' contributions

This work was carried out in collaboration between both authors. Author IOL proposed the algorithm. Author AOA developed, analysed and implemented the method. Authors AOA and IOL drafted the manuscript. Both authors read and approved the final manuscript.

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Abstract

A continuous one step Trigonometrically-fitted Third derivative method whose coefficients depend on the frequency and step size is derived using trigonometric basis function. The method obtained is use to solve standard problems with oscillating solutions. We also discuss the stability properties of the new method . Numerical result obtained via the implementation of the methods shows that the new method performs better than the one step Trigonometrically-fitted second derivative method proposed by Ngwane and Jator [1].

Keywords: Trigonometrically-fitted third derivative method; initial value problems; consistency; stability; initial value problem.

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1 Introduction

Mathematical modelling of real-life problems usually result into functional equations, e.g Ordinary Differential Equation(ODE), Partial Differntial Equation, Integral and Integro Differential equation, Stochastic Differential Equation and others. ODEs which normally arises in biological models, circuit theory models, circuit theory models, fluid and chemical kinetics models may or may not have exact solutions, thus a need for a numerical solution.

In this paper, we consider the of first order differential equation

$$y' = f(x, y), \; y(a) = y_0, \quad x \in [a, b], \tag{1.1}$$

with periodic or oscillating solutions where $f : \Re \times \Re^m \to \Re^m, \; y, \; y_0 \in \Re^m$

Several numerical methods (Brugnano and Trigiante [2], Jator, Akinfenwa and Yao [3], Odejide and Adeniran [4]) have been proposed for numerical solution of equation (1). Multiderivative method for solving systems of ODE was proposed by Obrenchkoff [5] and special cases of the Obrenchkoff were later proposed by Enright [6], Cash [7], Jia-Xiang, Jiao-Xun [8] and of recent Ehigie et al. [9]. And of these methods, the justification for including higher term in such method was clearly stated by Enright [6] which will include method with higher order, to obtain stability at infinity and to obtain a method with reasonable stability properties on the neighborhood of the origin. This class of Enright's schemes is a special class of the Obrenchkoff [5] methods which are found to be of order $p = k + 2$ for a k step method.

Ngwane and Jator [1] developed a continuous Trigonometrically-fitted Second Derivative Method whose coefficients depend on the frequency and stepsize, the method is constructed using trigonometric basis functions. Numerical experiments demonstrate the efficiency of the method for the numerical solution of ordinary differential equations with oscillatory solutions.

In this paper, we are motivated by the work of Ngwane and Jator [1] to develop a continuous third derivative multistep method.

2 Development of the Method

In this section, main objective is to develop a continuous trigonometrically fitted third derivative method which produce a discrete method as by-product. The method has the form

$$y_{n+1} = y_n + h[\beta_0(u) f_n + \beta_1(u) f_{n+1})] + h^2 \alpha_1(u) g_{n+1} + h^3 \gamma_1(u) e_{n+1} \tag{2.1}$$

where $u = kh$, $\beta_j(u)$, $\alpha_j(u)$, $\gamma_j(u)$, $j = 0, 1$, are coefficients that depend on the stepsize and frequency. y_{n+j} is the numerical approximation to the analytical solution $y(x_{n+j})$, where

$$\begin{aligned} f_{n+j} &= f(x_{n+j}, y_{n+j}), \\ g_{n+j} &= \frac{df(x, y(x))}{dx} |_{y_{n+j}}^{x_{n+j}}, \\ e_{n+j} &= \frac{d^2 f(x, y(x))}{dx^2} |_{y_{n+j}}^{x_{n+j}} \end{aligned}$$

In obtaining (2.2), we proceed by seeking approximation to exact solution $y(x)$ of the form

$$U(x) = a_0 + a_1 x + a_2 x^2 + a_3 \cos(kx) + a_4 \sin(kx) \tag{2.2}$$

where a_0, a_1, a_2, a_3, a_4 are coefficients that must be uniquely determined. we then impose that the interpolating function(2.3) coincide with analytical solution at point x_n to obtain the equation

$$U(x) = y_n \tag{2.3}$$

we also determine that the function (2.3) satisfies the differential equation (2.1) at the point x_{n+j}, $j = 0, 1$ to obtain the following set of equations

$$U'(x_{n+j}) = f_{n+j}, j = 0, 1,\ U''(x_{n+j}) = g_{n+j}, j = 1,\ U'''(x_{n+j}) = e_{n+j}, j = 1. \tag{2.4}$$

Equation (2.4) and (2.5) lead to a system of five linear equations which are solved to obtained the values of a_j, $j = 0(1)4$. The value of a_j's are then substituted into (2.6) to obtain our continuous trigonomtrically fitted third derivative of the form

$$U(x) = y_n + h[\beta_0(k,x)f_n + \beta_1(k,x)f_{n+1}] + h^2\alpha_1(k,x)g_{n+1} + h^3\gamma_1(k,x)e_{n+1} \tag{2.5}$$

k is the frequency, $\beta_0(k,x)$, $\beta_1(k,x)$, $\alpha_1(k,x)$, $\gamma_1(k,x)$ are continuous coefficients. The continuous coefficients in Equation (2.6) is used to generate the method of the form in Equation (2.2)

$$\begin{aligned}
\beta_0 &= \frac{-\cos(u) + 1 - \frac{u^2}{2}}{u(\sin(u) - u)} \\
\beta_1 &= \frac{\cos(u) + u\sin(u) - 1 - \frac{u^2}{2}}{u(\sin(u) - u)} \\
\alpha_1 &= \frac{-\cos(u) - \frac{u\sin(u)}{2} - 1}{u(\sin(u) - u)} \\
\gamma_1 &= \frac{-2 + 2u\sin(u) + 2\cos(u) - u^2\frac{\cos(u)}{2} - \frac{u^2}{2}}{u^3(\sin(u) - u)}
\end{aligned} \tag{2.6}$$

3 Error Analysis and Stability

3.1 Local truncation error

As $u \to 0$, the coefficient given by equation (3.1) are subject to heavy cancellations, thus Taylor series must be used (Simos [10]).

$$\begin{aligned}
\beta_0 &= \frac{1}{4} + \frac{1}{240}u^2 + \frac{1}{16800}u^4 + \frac{1}{2016000}u^6 - \frac{31}{13970880000}u^8 - \frac{89}{518918400000}u^{10} - \frac{2689}{762810048000000}u^{12} + \cdots \\
\beta_1 &= \frac{3}{4} - \frac{1}{240}u^2 - \frac{1}{16800}u^4 - \frac{1}{2016000}u^6 + \frac{31}{13970880000}u^8 + \frac{89}{518918400000}u^{10} + \frac{2689}{762810048000000}u^{12} + \cdots \\
\alpha_1 &= -\frac{1}{4} + \frac{1}{240}u^2 + \frac{1}{16800}u^4 + \frac{1}{2016000}u^6 - \frac{31}{13970880000}u^8 - \frac{89}{518918400000}u^{10} - \frac{2689}{762810048000000}u^{12} + \cdots \\
\gamma_1 &= \frac{1}{24} - \frac{1}{201600}u^4 - \frac{1}{9072000}u^6 - \frac{13}{9313920000}u^8 - \frac{17}{2724321600000}u^{10} + \frac{3533}{18307441152000000}u^{12} + \cdots
\end{aligned} \tag{3.1}$$

For practical computations when u is small, it is advisable to use the series expansion (3.1) . Thus the Local Truncation Error for method (2.6) subject to equation (3.1) is obtained as

$$\text{LTE} = -\frac{h^5}{2880}(k^2y^{(3)}(x_n) + y^{(5)}(x_n)) + O(h^6) \tag{3.2}$$

The method (2.6) specified by (3.1)is a fourth-order method.

3.2 Stability

Proposition 1. The trigonometrically fitted third derivative method (2.6) is applied to a test equation $y' = \lambda y$, $y'' = \lambda^2 y$ and $y''' = \lambda^3 y$, it yields

$$y_{n+1} = M(q;u)y_n \quad q = h\lambda, \quad u = kh \tag{3.3}$$

with

$$M(q;u) = \frac{1 - q\beta_1(u) - q^2\alpha_1(u) - q^3\gamma_1(u)}{1 + q\beta_0} \tag{3.4}$$

Proof. We begin by applying (2.6) to the test equation $y' = \lambda y$, $y'' = \lambda^2 y$ and $y''' = \lambda^3$ respectively, by letting $q = h\lambda$, $u = kh$, we obtain a linear equation which is used to solve for y_{n+1} with (3.4) as consequence.

Definition 1. *A region of stability is a region in the q-u plane, in which the rational function* $|M(q;u)| \leq 1$

Definition 2. *The method (2.6) is zero stable provided the root of the first characteristics polynomial have modulus less than or equal to unity and those of modulus unity are simple(Lambert [11], [12]).*

Definition 3. *Method (2.6) is consistent if it has order* $p > 1$ *(Fatunla [13])* The trigonometrically fitted third derivative method is consistient as it has order $p > 1$ and zero stable, hence convergent since Convergence =Zero stability + consistency

4 Implementation

In the spirit of Ngwane and Jator [1], method (2.6) is implemented to solve without predictors nor starting values. For instance , if we let $n = 0$ in (2.6), then y_1 is obtained on the sub interval $[x_0, x_1]$, as y_0 is obtained from the IVP, in a similar way, if we let $n = 1$, y_2 is obtained on the sub interval $[x_1, x_2]$, as y_1 is known from the previous computation and so on until we reach the final sun interval $[x_{N-1}, x_N]$. All computation were carried out , with the aid of MAPLE 17 software. For Linear problems, we solve (2.1) directly using the feature solve [], while for nonlinear problems, we solve (2.1) using the feature fsolve [].

5 Numerical Examples

In this section, we give numerical examples to illustrate the accuracy of our new method

Example 5.1: We consider an inhomogeneous IVP by Simos [10]

$$y'' = -100y + 99sin(x), \quad y(0) = 1, \; y'(0) = 11, \; x \in [0, 1000]$$

with the **exact** solution given as

$$y(x) = \cos(10x) + \sin(10x) + \sin(x).$$

The method in Simos [10] is of order 4 and exponential fitted, hence comparable to our method. we take $k = 1$

The numerical result for Example 5.1 were presented in Table 1. The problem was compared to other existing methods. Our new method displayed better accuracy within the range of integration.

Table 1. Comparism of error for Example 5.1

No. of steps N	Simos [10] $\max\|\|y_i - y(x_i)\|\|$	N	Ngwane and Jator [1] $\max\|\|y_i - y(x_i)\|\|$	Ours $\max\|\|y_i - y(x_i)\|\|$
4004	1.4×10^{-01}	1000	1.7×10^{-03}	9.9×10^{-04}
16004	1.1×10^{-02}	2000	2.4×10^{-04}	4.1×10^{-05}
32004	8.4×10^{-04}	8000	1.6×10^{-06}	7.0×10^{-07}
64004	5.5×10^{-06}	16000	1.0×10^{-07}	1.1×10^{-09}

Example 5.2: We also consider the moderately stiff problem

$$\begin{aligned} y' &= -y - 10z, \qquad & y(0) = 1 \\ z' &= -10y - z \qquad & z(0) = 1 \end{aligned}$$

exact solution $y(x) = e^{-x}\cos 10x, \;\; z(x) = e^{-x}\sin 10x$

The numerical result for Experiment 5.2 were presented in Table 2 below. The problem was compared to other existing method. Our new method displayed better accuracy within the range of integration. we take $k = 2$

Table 2. Numerical result for Example 5.2

No. of steps N	Ehigie et al. [9] $\max\|\|y_i - y(x_i)\|\|$	Ehigie et al. [9] $\max\|\|z_i - z(x_i)\|\|$	Ours $\max\|\|y_i - y(x_i)\|\|$	Ours $\max\|\|z_i - z(x_i)\|\|$
125	8.33×10^{-06}	1.32×10^{-6}	1.635×10^{-12}	4.795×10^{-12}
250	1.13×10^{-07}	1.36×10^{-08}	2.0×10^{-15}	1.6×10^{-14}
500	6.30×10^{-12}	8.19×10^{-12}	0	0

Example 5.3: Linear Kramarz problem. (Nguyen et al. [14])

$$y''(x) = \begin{pmatrix} 2498 & 4998 \\ -2499 & -4999 \end{pmatrix} y(t),\; y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix},\; y'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0 \le t \le 100$$

Exact solution: $y(x) = (2\cos(t), -\sin(t))^T$

We use this example to show the efficiency of our new method. Nguyen et al. [14] used the trigonometric implicit Runge-Kutta method and Ngwane and Jator [1] use the trigonometric second derivative method to solve the above linear Kramarz problem. Clearly, ours performs better. we take $k = 3$

Table 3. Comparism of error for Example 5.3

No. of steps N	Nguyen et al. [14] $\max\|\|y_i - y(x_i)\|\|$	N	Ngwane and Jator [1] $\max\|\|y_i - y(x_i)\|\|$	Ours $\max\|\|y_i - y(x_i)\|\|$
73	3.3×10^{-12}	10	1.3×10^{-15}	5.9×10^{-15}
142	9.0×10^{-12}	43	8.4×10^{-14}	1.1×10^{-15}
170	3.7×10^{-12}	80	7.1×10^{-15}	0

Example 5.4: Consider the given two-body problem which was solved by Ozawa [15].

$$y_1'' = -\frac{y_1}{r^3}, \;\; y_2'' = \frac{y_2}{r^3}, \;\; r = \sqrt{y_1^2 + y_2^2}$$

$$y_1(0) = 1 - e, \quad y_1'(0) = 0, \quad y_2(0) = 0, \quad y_2'(0) = \sqrt{\frac{1+e}{1-e}} \quad x \in [0, 5\pi]$$

wher, $0 \leq e < 1$ is an eccentricity. The exact solution of this equation problem is given as

$$\text{exact} : y_1(x) = \cos(m) - e, \quad y_2(x) = \sqrt{1 - e^2 \sin(m)}$$

where m is the solution of the Kepler's equation $k = x + e\sin(m)$, we take $k = 5$.

Table 4 contains the results obtained using the proposed method. These results were compared with the explicit singly diagonally implicit Runge-Kutta methods given in Ozawa [15]. In terms of accuracy, Table 4 clearly shows that Ours performs better than those in Ngwane and Jator [1] and Ozawa [15].

Table 4. Comparism of error for Example 5.4

No. of steps N	Ozawa [15] $\max\|\|y_i - y(x_i)\|\|$	N	Ngwane and Jator [1] $\max\|\|y_i - y(x_i)\|\|$	Ours $\max\|\|y_i - y(x_i)\|\|$
170	2.9×10^{-01}	150	1.2×10^{-02}	4.1×10^{-04}
225	7.8×10^{-03}	200	5.7×10^{-03}	1.0×10^{-04}
381	1.4×10^{-03}	300	3.1×10^{-04}	8.8×10^{-06}
680	1.7×10^{-04}	600	1.6×10^{-06}	7.0×10^{-08}
2144	1.9×10^{-06}	1600	4.9×10^{-10}	2.8×10^{-12}
6762	2.0×10^{-08}	3200	1.9×10^{-12}	5.5×10^{-14}

6 Conclusion

We have proposed a Trigonometrically fitted third derivative formula for solving oscillatory IVPs. The method is A-stable and hence can conviniently handle stiff IVPs. This method has the advantages of being self-starting, having good accuracy with order 4, and requires only three functions evaluation at each integration step.

Competing Interests

Authors have declared that no competing interests exist.

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