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Results of Existence Fixed Point for Integral Type Contractive Condition with w-distance

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Authors' contributions

This work was carried out in collaboration between the both authors. Author SJHG designed the study, carried out the theorems and wrote the first draft of the manuscript. Author KNA proved the theorms and genrate example. Both authors read and approved the final manuscript.

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Abstract

In this paper, we prove the existence of fixed point for mappings defined on complete metric spaces with w-distance that satisfying a general contractive inequality of integral type.

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1 Introduction

The first important result in fixed point theory is Banach's contraction principle. Branciari [1] established a fixed point result for an integral-type inequality, that is generalization of Banach's contraction principle. The concept of a w-distance on a metric space was introduced by Kada et al. [2] to generalize some important results in nonconvex minimizations. Literature abounds with several contractive conditions that have been employed by various researchers over the years to obtain different fixed point theorems. For various contractive definitions that have been employed, we refer our interested readers to Berinde [3, 4], Branciari [1], Rhoades [5] and Razani et al. [6, 7]. Branciari [1], Firouz et al. [8], Rhoades [5], Rouzkard et al. [9] and Takahashi et al. [10] used contractive conditions of integral type to extend Banach's fixed point theorem. We first introduced the concept of w-distance on a metric space. Then we present fixed point theorem for cotractions of integral type due to Baraciari. We also prove that w-distance contracion of integral type is true. In 1996, Kada and et al. in [2] for the first time introduced the concept of concept of w-distance on a metric space.

Definition 1.1. *Let X be a metric space with metric d. Then a function $p : X \times X \longrightarrow [0, \infty)$ is called w-distance in X if the following satisfy:*

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;
(2) for any $x \in X$, $p(x, .) \longrightarrow [0, \infty)$ is lower semicontinuous., that is, if $\{y_n\}$ is a sequence in X with $\lim_{n\to\infty} y_n = y \in X$, then $p(x, y) \leq liminf_{n\to\infty} p(x, y_n)$;
(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Let X be a metric space with metric d then d is a w-distance on X [2]. Branciari in [1] proved the following fixed point theorem.

Theorem 1.1. *Let (X, d) be a complete metric space,$c \in]0, 1[$, and let $f : X \to X$ a mapping such that for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt.$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable on each compact subset of $\mathbb{R}^+$,nonnegative, and for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0.$$

then f has a unique fixed point $a \in X$ such that $lim_{n\to\infty} f^n x = a$, for each $x \in X$.

2 Main Result

$\mathbb{N}$ will represent the set of natural numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}^+$ the set of nonnegative real numbers.

The main result of this paper is the following theorem; the proof is based on an argument similar to the one used by Baraciari [1].

Theorem 2.1. *Let (X, d) be a complete metric space,let p be a w-distance on X and ,$c \in]0, 1[$, and let $f : X \to X$ a mapping such that for each $x, y \in X$,*

$$\int_0^{p(fx,fy)} \varphi(t)dt \leq c \int_0^{p(x,y)} \varphi(t)dt. \quad (1)$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, nonnegative, and for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0. \quad (2)$$

then f has a unique fixed point $a \in X$ such that $lim_{n\to\infty} f^n x = a$, for each $x \in X$.

Proof. Let $x = x_o \in X$,and defined $x_n = f^n x$ for each integer $n \geq 1$. From (1), we have

$$\int_0^{p(x_n,x_{n+1})} \varphi(t)dt \leq c \int_0^{p(x_{n-1},x_n)} \varphi(t)dt \leq \cdots \leq c^n \int_0^{p(x_0,x_1)} \varphi(t)dt. \quad (3)$$

Taking the limit (3), as $n \to \infty$,by Lebesgue dominated convergence Theorem gives

$$\lim_{n\to\infty} \int_0^{p(x_n,x_{n+1})} \varphi(t)dt = 0 \quad (4)$$

which from (2), implies that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0 \quad (5)$$

We now show that $\{x_n\}$ is Cauchy. Suppose that $\{x_n\}$ is not p- Cauchy, that is

$$\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists m_\epsilon, n_\epsilon \in \mathbb{N}\, (m_\epsilon > n_\epsilon > N \,,\, p(x_m, x_n) \geq \epsilon)$$

We choose the sequences $\{m_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}}$ such that for $k \in \mathbb{N}$, m_k is minimal in the sense that $p(x_{m_k}, x_{n_k}) \geq \epsilon$, but $p(x_i, x_{n_k}) < \epsilon$ for each $i \in \{n_k + 1, \cdots, m_k - 1\}$. We have $p(x_{m_k}, x_{n_k}) \longrightarrow \epsilon+$ as $k \longrightarrow +\infty$, in fact by the triangular inequality and (5)

$$\begin{aligned} \epsilon \leq\ & p(x_{m_k}, x_{n_k}) \\ \leq\ & p(x_{m_k}, x_{m_k-1}) + p(x_{m_k-1}, x_{n_k}) \\ \leq\ & p(x_{m_k}, x_{m_k-1}) + \epsilon \longrightarrow \epsilon+ \end{aligned} \quad (6)$$

as $k \longrightarrow \infty$ further there exists $\mu \in \mathbb{N}$ such that for each natural number $k > \mu$, one has $p(x_{m_k+1}, x_{n_k+1}) < \epsilon$; because, if exists a subsequence $\{k_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that $p(x_{m_{k_j}+1}, x_{n_{k_j}+1}) \geq \epsilon$, than

$$\begin{aligned} \epsilon \leq\ & p(x_{m_{k_j}+1}, x_{n_{k_j}+1}) \leq p(x_{m_{k_j}+1}, x_{m_{k_j}}) \\ +\ & p(x_{m_{k_j}}, x_{n_{k_j}}) + p(x_{n_{k_j}}, x_{n_{k_j}+1}) \longrightarrow \epsilon \end{aligned} \quad (7)$$

as $j \longrightarrow \infty$ and from (1)

$$\int_0^{p(x_{m_{k_j}+1}, x_{n_{k_j}+1})} \varphi(t)dt \leq c \int_0^{p(x_{m_{k_j}}, x_{n_{k_j}})} \varphi(t)dt. \quad (8)$$

letting now $j \longrightarrow \infty$ in both sides of (8), we have $\int_0^\epsilon \varphi(t)dt \leq c \int_0^\epsilon \varphi(t)dt$ which is a contradiction being $c \in]0, 1[$ and the integral being positive. Therefore for a certain $\mu \in \mathbb{N}$ one has $p(x_{m_k}, x_{n_k}) < \epsilon$ for all $k > \mu$. Finally, we prove the stronger property that there that there exist a $h_\epsilon \in]0, \epsilon[$ and a N_ϵ such that for each $k > N_\epsilon$ we have $p(x_{m_k+1}, x_{n_k+1}) < \epsilon - h_\epsilon$; suppose the existence of a subsequence $\{k_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that $p(x_{m_{k_j}+1}, x_{n_{k_j}+1}) \longrightarrow \epsilon-$ as letting now $j \longrightarrow \infty$, then from

$$\int_0^{p(x_{m_{k_j}+1}, x_{n_{k_j}+1})} \varphi(t)dt \leq c \int_0^{p(x_{m_{k_j}}, x_{n_{k_j}})} \varphi(t)dt. \quad (9)$$

Again, letting $j \longrightarrow \infty$ in both sides of (9), we have the contradiction that $\int_0^\epsilon \varphi(t)dt \leq c \int_0^\epsilon \varphi(t)dt$. In conclusion, we can prove the Cauchy character of $\{x_n\}$. For each $k > N_\epsilon$ (N_ϵ as above)

$$\begin{aligned} \epsilon \leq\ & p(x_{m_k}, x_{n_k}) \leq p(x_{m_k}, x_{m_k+1}) \\ +\ & p(x_{m_k+1}, x_{n_k+1}) + p(x_{n_k+1}, x_{n_k}) \\ \leq\ & p(x_{m_k}, x_{m_k+1}) + (\epsilon - h_\epsilon) + p(x_{n_k+1}, x_{n_k+1}) \longrightarrow \epsilon - h_\epsilon \end{aligned} \quad (10)$$

thus $\epsilon \leq \epsilon - h_\epsilon$ which is a contradiction. This proves that $\{x_n\}$ is p-Cauchy, so it is Cauchy. Since (X, d) is a complete metric space, there exists a point $a \in X$ such that $a = \lim_{n\to\infty} x_n$; further a is a fixed point. For each $\epsilon > 0$ there exist $N_\epsilon \in \mathbb{N}$ such that $n > N_\epsilon$ implies $p(x_{N_\epsilon}, x_n) < \epsilon$ but $a = \lim_{n\to\infty} x_n$ and $p(x, .)$ is lower semi continuous thus

$$\begin{aligned} p(x_{N_\epsilon}, a) \leq\ & \liminf_{n\to\infty} p(x_{N_\epsilon}, x_n) \\ \leq\ & \epsilon \end{aligned}$$

there for $p(x_{N_\epsilon}, a) < \epsilon$, we put $\epsilon = 1/k, N_\epsilon = n_k$ and we have:

$$\lim_{k\to\infty} p(x_{n_k}, a) = 0. \tag{11}$$

In other hand, suppose $p(x_{n_k}, f(a))$ does not to 0 as $k \to \infty$, then there exist a subsequence $\{x_{n_{k_j+1}}\} \subseteq \{x_k + 1\}$ such that $p(x_{n_{k_j+1}}, f(a)) \geq \epsilon$ for a certain $\epsilon > 0$; thus we have the following contradiction

$$0 < \int_0^{\epsilon} \varphi(t)dt \leq \int_0^{p(x_{n_{k_j+1}}, f(a))} \varphi(t)dt \leq c \int_0^{p(x_{n_{k_j}}, a))} \varphi(t)dt \longrightarrow 0$$

as $j \to \infty$.Thus $\lim_{k\to\infty} p(x_{n_k}, f(a)) = 0$, but we have

$$p(x_{n_k}, f(a)) \leq p(x_{n_k}, x_{n_k+1}) + p(x_{n_k+1}, f(a))$$

thus

$$\lim_{k\to\infty} p(x_{n_k}, f(a)) = 0. \tag{12}$$

Now (11) and (12) implies $f(a) = a$.

Suppose there are two distinct fixed points $a, b \in X$ such that $f(a) = a$ and $f(b) = b$, then by (1) we have the following contradiction

$$0 < \int_0^{p(a,b)} \varphi(t)dt = \int_0^{p(f(a),f(b))} \varphi(t)dt \leq c \int_0^{p(a,b)} \varphi(t)dt < \int_0^{p(a,b)} \varphi(t)dt.$$

Finally, we have for each $x \in X$, $\lim_{n\longrightarrow\infty} x_n = lim_{n\to\infty} f^n x = a = fa$, for each $x \in X$. □

Remark 2.1. Theorem (2.1) is a generalization Theorem (1.1), letting $p = d$. The converse is not true because w- distance p can not be a metric.

Example 2.1. *Let $X := [0,1] \subseteq \mathbb{R}$ with metric induced by $\mathbb{R} : d(x,y) = |x - y|$, thus, since X is a closed subset of $\mathbb{R}$, it is a complete metric space. We have (X, d) is a complete metric space. Also, we define $p : X \times X \longrightarrow \mathbb{R}$ by $p(x,y) = y - x$ if $0 < x \leq y$ and $p(x,y) = 3x - 3y$ if $x > y > 0$ and $p(x,y) = 9$ if $x = 0$ or $y = 0$, this p is a w-distance on X. Letting $\varphi(t) = 1$ for each $t \geq 0$ in (1) and $c = \frac{1}{2}$. Also, we consider a mapping f from X into itself by $fx = \frac{x}{10}$ if $x \neq 0$,$f0 = 1$. For every $x, y \in X$, we have*

$$p(fx, fy) = \frac{1}{10} p(x,y) \leq cp(x,y) if \qquad x \neq 0, y \neq 0$$

$$p(f0, f0) = 0 \leq cp(0,0)$$

$$p(f0, fy) = 3 - fy \leq cp(0,y) \qquad if \qquad y \neq 0$$

$$p(fx, f0) = 1 - fx \leq cp(x,0) \qquad if \qquad x \neq 0.$$

But f don't have fixed point.

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Competing Interests

Authors have declared that no competing interests exist.

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