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Variants of Compatible Mappings in Menger Spaces

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, we introduce the notions of compatible mappings of type(R), type (K) and type (E) in Mengers paces and prove some common fixed point theorems for these mappings. In fact, we call these maps as variants of compatible mappings.

Keywords: Menger space; compatible mappings; compatible mappings of type (R); type (K); type (E).

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1 Introduction

The notion of probabilistic metric space as a generalization of metric space was introduced by Menger [1]. In Menger theory, the notion of probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In this note he explained how to replace the numerical distance between two points p and q by a function $\mathcal{F}(p,q,t)$ whose value $\mathcal{F}(p,q,t)$ at the real number t is interpreted as the probability that the distance

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between $p$and q is less than t. In fact the study of such spaces received an impetus with the pioneering work of Schweizerand Sklar [2]. The theory of probabilistic metric space is of paramount importance in Probabilistic Functional Analysis especially due to its extensive applications in random differential as well as random integral equations(see references) [3,4,5,6,7,8,9,10,11,12] and [13].

Now, we give preliminaries and basic definitions in Menger space which are useful in this paper.

Definition 1.1 [2]: A mapping $\mathcal{F}: \mathbb{R}^+ \to \mathbb{R}^+$ is called distribution function if it is non decreasing and left continuous with $\inf\{\mathcal{F}(t): t \in \mathbb{R}^+\} = 0$ and $\sup\{\mathcal{F}(t): t \in \mathbb{R}^+\} = 1$. We will denote the set of all distribution functions by $\mathcal{L}$.

Let $\mathcal{L}$ be the set of all distribution functions whereas $\mathcal{H}$ be the set of specific distribution function (Also known as Heaviside function) defined by

$$\mathcal{H}(t) = \begin{cases} 0, if t \leq 0 \\ 1, if t > 0. \end{cases}$$

Definition 1.2 [1]: A probabilistic metric space is a pair $(\mathbb{K}, \mathcal{F})$, where $\mathbb{K}$ is a nonempty set and $\mathcal{F}: \mathbb{K} \times \mathbb{K} \to \mathcal{L}$ is a mapping satisfying the following:

For all $p, q, r \in \mathbb{K}$ and $t, s \geq 0$,

$(p_1)\mathcal{F}(p, q, t) = 1$ if and only if $p = q$;
$(p_2)\mathcal{F}(p, q, 0) = 0$;
$(p_3)\mathcal{F}(p, q, t) = \mathcal{F}(q, p, t)$;
$(p_4)\mathcal{F}(p, q, t) = 1$ and $\mathcal{F}(q, r, s) = 1$, then $\mathcal{F}(p, r, t + s) = 1$.

Every metric space $(\mathbb{K}, d)$ can always be realized as a Probabilistiv metric space by $\mathcal{F}(p, q, t) = \mathcal{H}(t - d\,(p, q))$, for all $p, q \in \mathbb{K}$, where $\mathcal{H}$ be the set of specific distribution function defined in the definition 1.1 [2].

Probabilistic metric space offers a wider framework than that of the metric space and cover even wider statistical situations.

Definition 1.3 [2]: A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if for all $a, b, c \in [0,1]$,

(1) $\Delta(a, 1) = a, \Delta(0,0) = 0$;
(2) $\Delta(a, b) = \Delta(b, a)$;
(3) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a, d \geq b$;
(4) $\Delta(\Delta(a, b), c) = \Delta\big(a, \Delta(b, c)\big)$.

Example 1.4: The following are the four basic t-norms:

(i) The minimum t-norm: $\Delta_M(a, b) = \min\{a, b\}$.
(ii) The product t-norm: $\Delta_P(a, b) = ab$.
(iii) The Lukasiewicz t-norm: $\Delta_L(a, b) = max\,\{a + b - 1,0\}$.
(iv) The weakest t-norm, the drastic product:

$$\Delta_D(a, b) = \begin{cases} \min\{a, b\}\, if \max\{a, b\} = 1, \\ 0, \quad otherwise. \end{cases}$$

We have the following ordering in the above stated norms:

$$\Delta_D < \Delta_L < \Delta_P < \Delta_M .$$

Definition 1.5 [1]: A Menger space is a triplet $(\mathbb{K}, \mathcal{F}, \Delta)$, where $(\mathbb{K}, \mathcal{F})$ is a probabilistic metric space and Δ is a t-norm with the following condition:

For all $p, q, r \in \mathbb{K}$ and $t, s \geq 0$,

$$(p_5)\mathcal{F}(p, r, t + s) \geq \Delta(\mathcal{F}(p, q, t), \mathcal{F}(q, r, s)).$$

Example 1.6: Let $\mathbb{K} = \mathbb{R}, \Delta\,(a, b) = min\,(a, b)$, for all $a, b in\,[0,1]$ and

$$\mathcal{F}(p, q, t) = \begin{cases} \mathcal{H}(t), & if p \neq q \\ 1, & if p = q \end{cases}; \text{ where } \mathcal{H}(t) = \begin{cases} 0, if t \leq 0 \\ 1, if t > 0. \end{cases}$$

Then $(\mathbb{K}, \mathcal{F}, \Delta)$ is a Menger space.

Definition 1.7: A sequence $\{p_n\}$ in Menger space $(\mathbb{K}, \mathcal{F}, \Delta)$ is said to be:

(i) Convergent at a point $p \in \mathbb{K}$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon,\lambda}$ such that $\mathcal{F}(p_n, p, \epsilon) > 1 - \lambda$ for all $n \geq N_{\epsilon,\lambda}$.
(ii) Cauchy sequence in $\mathbb{K}$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon,\lambda}$ such that $\mathcal{F}(p_n, p_m, \epsilon) > 1 - \lambda$ for all $n, m \geq N_{\epsilon,\lambda}$.
(iii) Complete if every Cauchy sequence in $\mathbb{K}$ is convergent in $\mathbb{K}$.

In 1996, Jungck [14] introduce the notion of weakly commuting mappings.

Definition 1.8 [14]: Two self-mapping f_1 and g_1 of a Menger space $(\mathbb{K}, \mathcal{F}, \Delta)$ are said to be weakly commuting if $\mathcal{F}(f_1 g_1 p, g_1 f_1 p, t) \geq \mathcal{F}(f_1 p, g_1 p, t)$ for each $p \in \mathbb{K}$ and for each $t > 0$.

In 1982, Sessa [15] weakened the concept of commutativity to weakly commuting mappings. Afterwards, Jungck [16] enlarged the concept of weakly commuting mappings to compatible mappings.

In 1991, Mishra [17] introduced the notion of compatible mappings in the setting of probabilistic metric space.

Definition 1.9 [17]: Let $(\mathbb{K}, \mathcal{F}, \Delta)$be a Menger space such that the t $-$norm Δ is continuous and f_1, g_1 be mappings from $\mathbb{K}$ into itself. Then f_1 and g_1 are said to be compatible if $lim_{n\to\infty}\mathcal{F}(f_1 g_1 p_n, g_1 f_1 p_n, t) = 1$, whenever $\{p_n\}$ is a sequence in $\mathbb{K}$ such that $lim_{n\to\infty} f_1 p_n = lim_{n\to\infty} g_1 p_n = u_1$, for some $u_1 \in \mathbb{K}$.

Definition 1.10: Two self-mappings f_1 and g_1 on Menger space$(\mathbb{K}, \mathcal{F}, \Delta)$are said to be non-compatible if either

$$lim_{n\to\infty}\,\mathcal{F}(f_1 g_1 p_n, g_1 f_1 p_n\,, t) \text{ is non-existent or } lim_{n\to\infty}\,\mathcal{F}(f_1 g_1 p_n, g_1 f_1 p_n\,, t) \neq 1,$$

Whenever $\{p_n\}$ is a sequence in $\mathbb{K}$ such that $lim_{n\to\infty}\, f p_n = lim_{n\to\infty}\, g p_n = u_1$, for some $u_1 \in \mathbb{K}$.

Further, Singh and Jain [18] proved some fixed point theorems for weakly compatible maps in the setting of Menger space.

Definition 1.11 [18]: Two maps f_1 and g_1 are said to be weakly compatible if they commute at their coincidence points.

In 1999, Pant [19] introduced a new continuity condition in Menger space, known as reciprocal continuity as follows:

Definition 1.12 [19]: Let f_1 and $g_1$be self- mapping of a Menger space$(\mathbb{K}, \mathcal{F}, \Delta)$. Then f_1 and g_1 are said to be reciprocally continuous if $lim_{n\to\infty}\, f_1 g_1 p_n = f_1 r, lim_{n\to\infty}\, g_1 f_1 p_n = g_1 r$, whenever $\{p_n\}$ is a sequence in $\mathbb{K}$ such that $lim_{n\to\infty}\, f_1 p_n = lim_{n\to\infty}\, g_1 p_n = r$for some $r \in \mathbb{K}$.

Remark 1.13 [19]: If f_1 and g_1 are both continuous, then they are obviously reciprocally continuous, but the converse is not true. Moreover, common fixed point theorems for compatible pair of self-mappings satisfying contractive conditions, continuity of one of the mappings implies their reciprocal continuity, but not conversely.

In 2004, Rohan et al. [20] introduced the concept of compatible mappings of type (R) in a metric space as follows:

Definition 1.14 [20]: Let f_1 and g_1 be mappings from metric space $(\mathbb{K}, d)$ into itself. Then f_1 and g_1 are said to be compatible of type (R) if

$$\lim_{n\to\infty} d(f_1 g_1 p_n, g_1 f_1 p_n) = 0 \text{ and } \lim_{n\to\infty} d(f_1 f_1 p_n, g_1 g_1 p_n) = 0,$$

whenever $\{p_n\}$ is a sequence in $\mathbb{K}$ such that $\lim_{n\to\infty} f_1 p_n = \lim_{n\to\infty} g_1 p_n = u_1$, for some u_1 in $\mathbb{K}$.

In 2007, Singh and Singh et al. [21] introduced the concept of compatible mappings of type (E) in a metric space as follows:

Definition 1.15 [21]: Two self-mappings f_1 and g_1 of a metric space $(\mathbb{K}, d)$ are said to be compatible of type (E) if $\lim_{n\to\infty} f_1 f_1 p_n = \lim_{n\to\infty} f_1 g_1 p_n = g_1 u_1$ and $\lim_{n\to\infty} g_1 g_1 p_n = \lim_{n\to\infty} g_1 f_1 p_n = f_1 u_1$, whenever $\{p_n\}$ is a sequence in $\mathbb{K}$ such that $\lim_{n\to\infty} f_1 p_n = u_1$ for some t in $\mathbb{K}$.

In 2014, Jha et al. [22] introduced the concept of compatible mappings of type (K) in a metric space as follows:

Definition 1.16 [22]: Let f_1 and g_1 be mappings from metric space $(\mathbb{K}, d)$ into itself. Then f_1 and g_1 are said to be compatible of type (K) if

$$\lim_{n\to\infty} d(f_1 f_1 p_n, g_1 u_1) = 0 \text{ and } \lim_{n\to\infty} d(g_1 g_1 p_n, f_1 u_1) = 0,$$

whenever $\{p_n\}$ is a sequence in $\mathbb{K}$ such that $\lim_{n\to\infty} f_1 p_n = \lim_{n\to\infty} g_1 p_n = u_1$, for some u_1 in $\mathbb{K}$.

2 Properties of Variants of Compatible Mappings

Now we present the notions of variants of compatible mappings in the context of a Menger space.

Definition 2.1: Let $\mathcal{S}$ and $\mathcal{T}$ are two self-mapping on Menger space $(\mathbb{K}, \mathcal{F}, \Delta)$. Then $\mathcal{S}$ and $\mathcal{T}$ are said to be:

1. Compatible of type (R) if $lim_{n\to\infty} \mathcal{F}(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n, t_1) = 1$, and $\lim_{n\to\infty} \mathcal{F}(\mathcal{S}\mathcal{S}x_n, \mathcal{T}\mathcal{T}x_n, t_1) = 1$, whenever a sequence $\{x_n\}$ in $\mathbb{K}$ satisfying $lim_{n\to\infty} \mathcal{S}x_n = lim_{n\to\infty} \mathcal{T}x_n = u_1$, where $u_1 \in \mathbb{K}$, $\forall t_1 > 0$.

2. Compatible of type (K) if $\lim_{n\to\infty} \mathcal{F}(\mathcal{S}\mathcal{S}x_n, \mathcal{T}u_1, t_1) = 1$ and $\lim_{n\to\infty} \mathcal{F}(\mathcal{T}\mathcal{T}x_n, \mathcal{S}u_1, t_1) = 1$, whenever a sequence $\{x_n\}$ in $\mathbb{K}$ satisfying $lim_{n\to\infty} \mathcal{S}x_n = lim_{n\to\infty} \mathcal{T}x_n = u_1$, where u_1 in $\mathbb{K}$.

3. Compatible of type (E) if $\lim_{n\to\infty} \mathcal{S}\mathcal{S}x_n = \lim_{n\to\infty} \mathcal{S}\mathcal{T}x_n = \mathcal{T}u_1$ and $lim_{n\to\infty} \mathcal{T}\mathcal{T}x_n = lim_{n\to\infty} \mathcal{T}\mathcal{S}x_n = \mathcal{S}u_1$, whenever a sequence $\{x_n\}$ is in $\mathbb{K}$ satisfying $lim_{n\to\infty} \mathcal{S}x_n = lim_{n\to\infty} \mathcal{T}x_n = u_1$, where u_1 in $\mathbb{K}$.

Proposition 2.1: Let $\mathcal{S}$ and $\mathcal{T}$ are two compatible mappings of type (R) self maps of a Menger space $(\mathbb{K}, \mathcal{F}, \Delta)$. If $\mathcal{S}u_1 = \mathcal{T}u_1$, for some $u_1 \in \mathbb{K}$, then $\mathcal{S}\mathcal{T}u_1 = \mathcal{S}\mathcal{S}u_1 = \mathcal{T}\mathcal{T}u_1 = \mathcal{T}\mathcal{S}u_1$.

Proof: Suppose that $\{x_n\}$ is a sequence in $\mathbb{K}$ defined by $x_n = u_1, n = 1,2, ...$ for some $u_1 \in \mathbb{K}$ and $\mathcal{S}u_1 = \mathcal{T}u_1$.Thenwe have $\mathcal{S}x_n, \mathcal{T}x_n \to \mathcal{S}u_1$ as $n \to \infty$. Since $\mathcal{S}$and $\mathcal{T}$ are compatible of type (R), we have

$$\mathcal{F}(\mathcal{S}\mathcal{T}u_1, \mathcal{T}\mathcal{S}u_1, t_1) = lim_{n\to\infty}\mathcal{F}(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n, t_1) = 1.$$

Hence we have $\mathcal{S}\mathcal{T}u_1 = \mathcal{S}\mathcal{S}u_1$. Therefore, since $\mathcal{S}u_1 = \mathcal{T}u_1$, we have $\mathcal{S}\mathcal{T}u_1 = \mathcal{S}\mathcal{S}u_1 = \mathcal{T}\mathcal{T}u_1 = \mathcal{T}\mathcal{S}u_1$.

Proposition 2.2: Let $\mathcal{S}$ and $\mathcal{T}$ are two compatible mappings of type (R) self maps of Menger space $(\mathbb{K}, \mathcal{F}, \Delta)$. Consider that $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = u_1$, where$u_1$ in $\mathbb{K}$. Then

(a) $\lim_{n\to\infty} \mathcal{T}\mathcal{S}x_n = \mathcal{S}u_1$ if $\mathcal{S}$ is continuous at u_1.
(b) $\lim_{n\to\infty} \mathcal{S}\mathcal{T}x_n = \mathcal{T}u_1$ if $\mathcal{T}$ is continuous at u_1.
(c) $\mathcal{S}\mathcal{T}u_1 = \mathcal{T}\mathcal{S}u_1$ and $\mathcal{S}u_1 = \mathcal{T}u_1$ if $\mathcal{S}$and $\mathcal{T}$are continuous at u_1.

Proof: (a) Suppose that $\mathcal{S}$ is continuous at u_1. Since $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = u_1$ whereu_1 in $\mathbb{K}$, we have $\mathcal{S}\mathcal{S}x_n, \mathcal{S}\mathcal{T}x_n \to \mathcal{S}u_1$ as $n \to \infty$.Then by given condition

$$lim_{n\to\infty}\mathcal{F}(\mathcal{T}\mathcal{S}x_n, \mathcal{S}u_1, t_1) = lim_{n\to\infty}\mathcal{F}(\mathcal{T}\mathcal{S}x_n, \mathcal{S}\mathcal{T}x_n, t_1) = 1.$$

Therefore, $\lim_{n\to\infty} \mathcal{T}\mathcal{S}x_n = \mathcal{S}u_1$.

(b) Suppose that $\mathcal{T}$ is continuous at u_1.Since $lim_{n\to\infty} \mathcal{S}x_n = lim_{n\to\infty} \mathcal{T}x_n = u_1$, where in $\mathbb{K}$, we have $\mathcal{T}\mathcal{S}x_n, \mathcal{T}\mathcal{T}x_n \to \mathcal{T}u_1$as $n \to \infty$.Then by given condition

$$\lim_{n\to\infty}\mathcal{F}(\mathcal{S}\mathcal{T}x_n, \mathcal{T}u_1, t_1) = \lim_{n\to\infty}\mathcal{F}(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n, t_1) = 1.$$

Therefore, $\lim_{n\to\infty} \mathcal{S}\mathcal{T}x_n = \mathcal{T}u_1$.

(c) Easily follow by Proposition 4.1, $\mathcal{S}\mathcal{T}u_1 = \mathcal{S}\mathcal{S}u_1 = \mathcal{T}\mathcal{T}u_1 = \mathcal{T}\mathcal{S}u_1$.

Proposition 2.3: Let $\mathcal{S}$ and $\mathcal{T}$ are two compatible mappings of type (E) on a Mengerspace $(\mathbb{K}, \mathcal{F}, \Delta)$ into itself.

If one of $\mathcal{S}$ and $\mathcal{T}$ be continuous and
$lim_{n\to\infty} \mathcal{S}x_n = u_1$, where $u_1 \in \mathbb{K}$. Then the following hold

(a) $\mathcal{S}u_1 = \mathcal{T}u_1, \lim_{n\to\infty} \mathcal{S}\mathcal{S}x_n = \lim_{n\to\infty} \mathcal{S}\mathcal{T}x_n = \lim_{n\to\infty} \mathcal{T}\mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}\mathcal{T}x_n$.
(b) If we can find$v_1 \in \mathbb{K}$ such that $\mathcal{S}v_1 = \mathcal{T}v_1 = u_1$,we have $\mathcal{S}\mathcal{T}v_1 = \mathcal{T}\mathcal{S}v_1$.

Lemma 2.1 [18]: Let $(\mathbb{K}, \mathcal{F}, \Delta)$be aMenger space.If there exists a k in (0,1) such that

$$\mathcal{F}(p, q, kt) \geq \mathcal{F}(p, q, t) \text{for all} p, q \in \mathbb{K} \text{ and } t > 0, \text{ then } p = q.$$

Lemma 2.2 [18]: Let $\{p_n\}$ be a sequence in a Menger space$(\mathbb{K}, \mathcal{F}, \Delta)$with continuous t -norm Δ and $\Delta(t, t) \geq t$. If there exists a k in (0,1) such that

$$\mathcal{F}(p_n, p_{n+1}, kt) \geq \mathcal{F}(p_{n-1}, p_n, t), \text{ then } \{p_n\} \text{ is a Cauchy sequence in } \mathbb{K}.$$

3 Main Results

Now we prove our main theorems in Menger spaces.

Theorem 3.1: Let $\mathcal{A}, \mathcal{S}, \mathcal{B}$ and $\mathcal{T}$are four self maps on a complete Menger space $(\mathbb{K}, \mathcal{F}, \Delta)$ such that

(3.1) $\mathcal{T}(\mathbb{K}) \subset \mathcal{A}(\mathbb{K})$, $\mathcal{S}(\mathbb{K}) \subset \mathcal{B}(\mathbb{K})$;

(3.2) $$\mathcal{F}(\mathcal{S}z, \mathcal{T}w, qt_1) \geq min\begin{Bmatrix} \mathcal{F}(\mathcal{A}z, \mathcal{B}w, t_1), \mathcal{F}(\mathcal{A}z, \mathcal{S}z, t_1), \\ \mathcal{F}(\mathcal{B}w, \mathcal{T}w, t_1), \mathcal{F}(\mathcal{S}z, \mathcal{B}w, \alpha t_1), \\ \mathcal{F}(\mathcal{A}z, \mathcal{T}w, (2-\alpha)t_1) \end{Bmatrix}$$

hold for all z, w in $\mathbb{K}$, where $\alpha \in (0,2), t_1 > 0$,

(3.3) one of the map$\mathcal{A}, \mathcal{S}, \mathcal{B}$ and $\mathcal{T}$ be continuous.

Suppose the pairs $(\mathcal{B}, \mathcal{T})$and $(\mathcal{A}, \mathcal{S})$are compatible of type (R).

Then$u_1 = \mathcal{B}u_1 = \mathcal{T}u_1 = \mathcal{A}u_1 = \mathcal{S}u_1$, where u_1 is a unique in $\mathbb{K}$.

Proof: Since $\mathcal{S}(\mathbb{K}) \subset \mathcal{B}(\mathbb{K})$. Now consider a point $z_0 \in \mathbb{K}$, we have a point$z_1 \in \mathbb{K}$ such that $\mathcal{S}z_0 = \mathcal{B}z_1 = w_0$, for$z_1$, we can find a point$z_2 \in \mathbb{K}$satisfying $\mathcal{T}z_1 = \mathcal{A}z_2 = w_1$. Similarly we have a sequence $\{w_n\}$ in $\mathbb{K}$ satisfying

$$w_{2n} = \mathcal{S}z_{2n} = \mathcal{B}z_{2n+1};$$
$$w_{2n+1} = \mathcal{T}z_{2n+1} = \mathcal{A}z_{2n+2};$$

Now we prove that $\{w_n\}$ is Cauchy sequence in $\mathbb{K}$.

On setting $z = z_{2n}$, $w = z_{2n+1}$, $\alpha = 1 - \beta$with $\beta \in (0,1)$in inequality (3.2), we have

$$\mathcal{F}(\mathcal{S}z_{2n}, \mathcal{T}z_{2n+1}, qt_1) \geq min\begin{Bmatrix} \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{B}z_{2n+1}, t_1), \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{S}z_{2n}, t_1), \\ \mathcal{F}(\mathcal{B}z_{2n+1}, \mathcal{T}z_{2n+1}, t_1), \\ \mathcal{F}(\mathcal{S}z_{2n}, \mathcal{B}z_{2n+1}, (1-\beta)t_1) \\ \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{T}z_{2n+1}, (1+\beta)\, t_1) \end{Bmatrix}$$

$$\mathcal{M}(w_{2n}, w_{2n+1}, qt_1) \geq min\begin{Bmatrix} \mathcal{F}(w_{2n-1}, w_{2n}, t_1), \mathcal{F}(w_{2n-1}, w_{2n}, t_1), \\ \mathcal{F}(w_{2n}, w_{2n+1}, t_1), \\ \mathcal{F}(w_{2n}, w_{2n}, (1-\beta)t_1), \\ \mathcal{F}(w_{2n-1}, w_{2n+1}, (1+\beta)\, t_1) \end{Bmatrix}$$

$$\mathcal{F}(w_{2n}, w_{2n+1}, qt_1) \geq min\begin{Bmatrix} \mathcal{F}(w_{2n-1}, w_{2n}, t_1), \\ \mathcal{F}(w_{2n}, w_{2n+1}, t_1), 1, \\ \mathcal{F}(w_{2n-1}, w_{2n+1}, (1+\beta)\, t_1) \end{Bmatrix}$$

$$\geq min\begin{Bmatrix} \mathcal{F}(w_{2n-1}, w_{2n}, t_1), \mathcal{F}(w_{2n}, w_{2n+1}, t_1), \\ \mathcal{F}(w_{2n-1}, w_{2n+1}, (1+\beta)\, t_1) \end{Bmatrix}$$

$$\geq min\begin{Bmatrix} \mathcal{F}(w_{2n-1}, w_{2n}, t_1), \mathcal{F}(w_{2n}, w_{2n+1}, t_1), \\ \mathcal{F}(w_{2n-1}, w_{2n}, t_1), \mathcal{F}(w_{2n}, w_{2n+1}, \beta t_1) \end{Bmatrix}$$

$$\geq min\begin{Bmatrix} \mathcal{F}(w_{2n-1}, w_{2n}, t_1), \mathcal{F}(w_{2n}, w_{2n+1}, t_1), \\ \mathcal{F}(w_{2n}, w_{2n+1}, \beta t_1) \end{Bmatrix}$$

SinceΔ is continuous, letting $\beta \to 1$ we obtain

$$\mathcal{F}(w_{2n}, w_{2n+1}, qt_1) \geq min\begin{Bmatrix} \mathcal{F}(w_{2n-1}, w_{2n}, t_1), \\ \mathcal{F}(w_{2n}, w_{2n+1}, t_1) \end{Bmatrix}$$

$$= min\{\mathcal{F}(w_{2n-1}, w_{2n}, t_1), \mathcal{F}(w_{2n}, w_{2n+1}, t_1)\},$$

Hence $\mathcal{M}(w_{2n}, w_{2n+1}, qt_1) \geq min\{\mathcal{F}(w_{2n-1}, w_{2n}, t_1), \mathcal{F}(w_{2n}, w_{2n+1}, t_1)\}$

Similarly

$$\mathcal{F}(w_{2n+1}, w_{2n+2}, qt_1) \geq min\{\mathcal{F}(w_{2n}, w_{2n+1}, t_1), \mathcal{F}(w_{2n+1}, w_{2n+2}, t_1)\}$$

So for all n, we have

$$\mathcal{F}(w_n, w_{n+1}, qt_1) \geq min\{ \mathcal{F}(w_{n-1}, w_n, t_1), \mathcal{F}(w_n, w_{n+1}, t_1)\}.$$

Consequently, we have

$$\mathcal{F}(w_n, w_{n+1}, t_1) \geq min\{ \mathcal{F}(w_{n-1}, w_n, \frac{t_1}{q}), \mathcal{F}(w_n, w_{n+1}, \frac{t_1}{q})\}.$$

By repeated application of above inequality, we get

$$\mathcal{F}(w_n, w_{n+1}, t_1) \geq min\{ \mathcal{F}(w_{n-1}, w_n, \frac{t_1}{q}), \mathcal{F}(w_n, w_{n+1}, \frac{t_1}{q^m})\}.$$

Since $\mathcal{F}(w_n, w_{n+1}, \frac{t_1}{q^m}) \to 1$ as $m \to \infty$, it follows that

$$\mathcal{F}(w_n, w_{n+1}, qt_1) \geq \mathcal{F}(w_{n-1}, w_n, t_1), \text{ for all } n \in \mathbb{N}.$$

By Lemma 2.2, $\{w_n\}$ be a Cauchy sequence in $\mathbb{K}$ and hence it converges to $u_1 \in \mathbb{K}$, then the subsequence $\{\mathcal{S}z_{2n}\}$, $\{\mathcal{B}z_{2n+1}\}$, $\{\mathcal{T}z_{2n+1}\}$ and $\{\mathcal{A}z_{2n}\}$ of $\{w_n\}$ also converges to u_1.

Suppose $\mathcal{A}$ is continuous. Now by Proposition 2.2 and $(\mathcal{A}, \mathcal{S})$ are compatible of type (R), $\mathcal{A}\mathcal{A}z_{2n}$ and $\mathcal{S}\mathcal{A}z_{2n}$ converges to $\mathcal{A}u_1$ as $n \to \infty$.

We claim that $u_1 = \mathcal{A}u_1$.

On putting $z = \mathcal{A}z_{2n}$ and $w = z_{2n+1}, \alpha = 1$ in inequality (3.2), we have

$$\mathcal{F}(\mathcal{S}\mathcal{A}z_{2n}, \mathcal{T}z_{2n+1}, qt_1) \geq min\begin{Bmatrix} \mathcal{F}(\mathcal{A}\mathcal{A}z_{2n}, \mathcal{B}z_{2n+1}, t_1), \mathcal{F}(\mathcal{A}\mathcal{A}z_{2n}, \mathcal{S}\mathcal{A}z_{2n}, t_1), \\ \mathcal{F}(\mathcal{B}z_{2n+1}, \mathcal{T}z_{2n+1}, t_1), \mathcal{F}(\mathcal{S}\mathcal{A}z_{2n}, \mathcal{B}z_{2n+1}, t_1), \\ \mathcal{F}(\mathcal{A}\mathcal{A}z_{2n}, \mathcal{T}z_{2n+1}, t_1) \end{Bmatrix}$$

Letting $n \to \infty$, we get

$$\mathcal{F}(\mathcal{A}u_1, u_1, qt_1) \geq min\begin{Bmatrix} \mathcal{F}(\mathcal{A}u_1, u_1, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{A}u_1, t_1), \\ \mathcal{F}(u_1, u_1, t_1), \mathcal{F}(\mathcal{A}u_1, u_1, t_1), \\ \mathcal{F}(\mathcal{A}u_1, u_1, t_1) \end{Bmatrix}$$

$\mathcal{F}(\mathcal{A}u_1, u_1, qt_1) \geq \mathcal{F}(\mathcal{A}u_1, u_1, t_1)$.

Lemma 2.1gives, $\mathcal{A}u_1 = u_1$.

Next we claim that $\mathcal{S}u_1 = u_1$.

Putting $z = u_1$ and $w = z_{2n+1}, \alpha = 1$ in inequality (3.2), we have

$$\mathcal{F}(\mathcal{S}u_1, \mathcal{T}z_{2n+1}, qt_1) \geq min\begin{Bmatrix} \mathcal{F}(\mathcal{A}u_1, \mathcal{B}z_{2n+1}, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{S}u_1, t_1), \\ \mathcal{F}(\mathcal{B}z_{2n+1}, \mathcal{T}z_{2n+1}, t_1), \\ \mathcal{F}(\mathcal{S}u_1, \mathcal{B}z_{2n+1}, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{T}z_{2n+1}, t_1) \end{Bmatrix}$$

Letting$n \to \infty$, we obtain

$$\mathcal{F}(\mathcal{S}u_1, u_1, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(u_1, u_1, t_1), \mathcal{F}(u_1, \mathcal{S}u_1, t_1), \\ \mathcal{F}(u_1, u_1, t_1), \mathcal{F}(\mathcal{S}u_1, u_1, t_1), \\ \mathcal{F}(u_1, u_1, t_1) \end{Bmatrix}$$

we have $\mathcal{F}(\mathcal{S}u_1, u_1, qt_1) \geq \mathcal{F}(\mathcal{S}u_1, u_1, t_1)$.

Lemma 2.1 gives, $\mathcal{S}u_1 = u_1$.

Since $\mathcal{S}(\mathbb{K}) \subset \mathcal{B}(\mathbb{K})$ and hence we can find a point v_1 in $\mathbb{K}$ satisfying$u_1 = \mathcal{S}u_1 = \mathcal{B}v_1$.

We claim that $u_1 = \mathcal{T}v_1$.

On setting$z = u_1$ and $w = v_1, \alpha = 1$ in inequality (3.2), we obtain

$$\mathcal{F}(u_1, \mathcal{T}v_1, qt_1) = \mathcal{F}(\mathcal{S}u_1, \mathcal{T}v_1, kt_1) \geq min \begin{Bmatrix} \mathcal{F}(\mathcal{A}u_1, \mathcal{B}v_1, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{S}u_1, t_1), \\ \mathcal{F}(\mathcal{B}v_1, \mathcal{T}v_1, t_1) \\ , \mathcal{F}(\mathcal{S}u_1, \mathcal{B}v_1, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{T}v_1, t_1) \end{Bmatrix}$$

i.e., $\mathcal{F}(u_1, \mathcal{T}v_1, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(u_1, u_1, t_1), \mathcal{F}(u_1, u_1, t_1), \mathcal{F}(u_1, \mathcal{T}v_1, t_1), \\ \mathcal{F}(u_1, u_1, t_1), \mathcal{F}(u_1, v_1, t_1) \end{Bmatrix}$

i.e., $\mathcal{F}(u_1, \mathcal{T}v_1, qt_1) \geq \mathcal{F}(u_1, \mathcal{T}v_1, t_1)$.

By Lemma 2.1, we get $u_1 = \mathcal{T}v_1$.

Since $\mathcal{B}$ and $\mathcal{T}$ are compatible of type (R) and $\mathcal{B}v_1 = \mathcal{T}v_1 = u_1$, by Proposition 2.1, $\mathcal{B}\mathcal{T}v_1 = \mathcal{T}\mathcal{B}v_1$ and hence $\mathcal{B}u_1 = \mathcal{B}\mathcal{T}v_1 = \mathcal{T}\mathcal{B}v_1 = \mathcal{T}u_1$. Also, we obtain

$$\mathcal{F}(u_1, \mathcal{B}u_1, qt_1) = \mathcal{F}(\mathcal{S}u_1, \mathcal{T}u_1, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(\mathcal{A}u_1, \mathcal{B}u_1, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{S}u_1, t_1), \\ \mathcal{F}(\mathcal{B}u_1, \mathcal{T}u_1, t_1) \\ , \mathcal{F}(\mathcal{S}u_1, \mathcal{B}u_1, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{T}u_1, t_1) \end{Bmatrix}$$

we get $\mathcal{F}(u_1, \mathcal{B}u_1, qt_1) \geq \mathcal{F}(u_1, \mathcal{B}u_1, t_1)$.

By Lemma 2.1, we get $u_1 = \mathcal{B}u_1$. Hence $u_1 = \mathcal{B}u_1 = \mathcal{T}u_1 = \mathcal{A}u_1 = \mathcal{S}u_1$.

Now suppose$\mathcal{S}$ is continuous given $\mathcal{A}$ and $\mathcal{S}$ are compatible of type (R), by Proposition 2.2, $\mathcal{S}\mathcal{S}z_{2n}$ and $\mathcal{S}z_{2n}$ converges to $\mathcal{S}u_1$as$n \to \infty$.

We claim that $u_1 = \mathcal{S}u_1$.

Putting $z = \mathcal{S}z_{2n}$ and $w = z_{2n+1}, \alpha = 1$ in inequality (3.2) ,we have

$$\mathcal{F}(\mathcal{S}\mathcal{S}z_{2n}, \mathcal{T}z_{2n+1}, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(\mathcal{A}\mathcal{S}z_{2n}, \mathcal{B}z_{2n+1}, t_1), \mathcal{F}(\mathcal{A}\mathcal{S}z_{2n}, \mathcal{S}\mathcal{S}z_{2n}, t_1), \\ \mathcal{F}(\mathcal{B}z_{2n+1}, \mathcal{T}z_{2n+1}, t_1), \mathcal{F}(\mathcal{S}\mathcal{S}z_{2n}, \mathcal{B}z_{2n+1}, t_1), \\ \mathcal{F}(\mathcal{A}\mathcal{S}z_{2n}, \mathcal{T}z_{2n+1}, t_1) \end{Bmatrix}$$

Letting $n \to \infty$ we get

we get $\mathcal{F}(\mathcal{S}u_1, u_1, qt_1) \geq \mathcal{F}(\mathcal{S}u_1, u_1, t_1)$.

By Lemma 2.1, we $\mathcal{S}u_1 = u_1$.

Since $\mathcal{S}(\mathcal{K}) \subset \mathcal{B}(\mathcal{K})$ and hence we can find a point $p_1 \in \mathcal{K}$ satisfying$u_1 = \mathcal{S}u_1 = \mathcal{B}p_1$.

We claim that $u_1 = \mathcal{T}p_1$.

Putting $z = \mathcal{S}z_{2n}$ and $w = p_1, \alpha = 1$ in inequality (3.2), we obtain

$$\mathcal{F}(\mathcal{S}\mathcal{S}z_{2n}, \mathcal{T}p_1, qt_1) \geq min\begin{Bmatrix} \mathcal{F}(\mathcal{A}\mathcal{S}z_{2n}, \mathcal{B}p_1, t_1), \mathcal{F}(\mathcal{A}\mathcal{S}z_{2n}, \mathcal{S}\mathcal{S}z_{2n}, t_1), \\ \mathcal{F}(\mathcal{B}p_1, \mathcal{T}p_1, t_1), \mathcal{F}(\mathcal{S}\mathcal{S}z_{2n}, \mathcal{B}p_1, t_1), \\ \mathcal{F}(\mathcal{A}\mathcal{S}z_{2n}, \mathcal{T}p_1, t_1) \end{Bmatrix}$$

Letting $n \to \infty$ we have

$$\mathcal{F}(u_1, \mathcal{T}p_1, qt_1) \geq min\{\mathcal{F}(u_1, u_1, t_1), \mathcal{F}(u_1, \mathcal{T}p_1, t_1)\}$$

we get $\mathcal{F}(u_1, \mathcal{T}p_1, qt_1) \geq \mathcal{F}(u_1, \mathcal{T}p_1, t_1)$.

By Lemma 2.1, we get $u_1 = \mathcal{T}p_1$.

Since $\mathcal{B}$ and $\mathcal{T}$ are compatible of type (R) and $\mathcal{B}p_1 = \mathcal{T}p_1 = u_1$, by Proposition 2.2, $\mathcal{B}\mathcal{T}p_1 = \mathcal{T}\mathcal{B}p_1$ and hence $\mathcal{B}u_1 = \mathcal{B}\mathcal{T}p_1 = \mathcal{T}\mathcal{B}p_1 = \mathcal{T}u_1$.

We claim that $u_1 = \mathcal{T}u_1$.

Putting $z = z_{2n}$ and $w = u_1, \alpha = 1$ in inequality (3.2), we have

$$\mathcal{F}(\mathcal{S}z_{2n}, \mathcal{T}u_1, qt_1) \geq min\begin{Bmatrix} \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{B}u_1, t_1), \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{S}z_{2n}, t_1), \\ \mathcal{F}(\mathcal{B}u_1, \mathcal{T}u_1, t_1), \\ , \mathcal{F}(\mathcal{S}z_{2n}, \mathcal{B}u_1, t_1), \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{T}u_1, t_1) \end{Bmatrix}$$

Letting $n \to \infty$, we have $\mathcal{F}(u_1, \mathcal{T}u_1, qt_1) \geq min\{\mathcal{F}(u_1, \mathcal{T}u_1, t_1), \mathcal{F}(u_1, u_1, t_1)\}$

we have$\mathcal{F}(u_1, \mathcal{T}u_1, qt_1) \geq \mathcal{F}(u_1, \mathcal{T}u_1, t_1)$.

By Lemma 2.1, we $\mathcal{T}u_1 = u_1$.

Since $\mathcal{T}(\mathbb{K}) \subset \mathcal{A}(\mathbb{K})$ and hence we can find a point $x_1 \in \mathbb{K}$ satisfying$u_1 = \mathcal{S}u_1 = \mathcal{A}x_1$.

We show that $u_1 = \mathcal{S}x_1$.

Putting $z = x_1$ and $w = u_1, \alpha = 1$ in inequality (3.2), we obtain

$$\mathcal{F}(\mathcal{S}x_1, u_1, qt_1) = \mathcal{F}(\mathcal{S}x_1, \mathcal{T}u_1, kt_1) \geq min\begin{Bmatrix} \mathcal{F}(\mathcal{A}x_1, \mathcal{B}u_1, t_1), \mathcal{F}(\mathcal{A}x_1, \mathcal{S}x_1, t_1), \\ \mathcal{F}(\mathcal{B}u_1, \mathcal{T}u_1, t_1), \mathcal{F}(\mathcal{S}x_1, \mathcal{B}x_1, t_1), \\ \mathcal{F}(\mathcal{A}x_1, \mathcal{T}u_1, t_1) \end{Bmatrix}$$

$$= min\{\mathcal{F}(u_1, u_1, t_1), \mathcal{F}(u_1, \mathcal{S}x_1, t_1)\}$$

we obtain $\mathcal{F}(\mathcal{S}x_1, u_1, qt_1) \geq \mathcal{F}(u_1, \mathcal{S}x_1, t_1)$.

Lemma 2.1 gives, $u_1 = \mathcal{S}x_1$.

Since $\mathcal{S}$ and $\mathcal{A}$ are compatible of type (R) and $\mathcal{S}x_1 = \mathcal{A}x_1 = u_1$, by Proposition 2.1, $\mathcal{A}\mathcal{S}x_1 = \mathcal{S}\mathcal{A}x_1$ and hence $\mathcal{A}u_1 = \mathcal{A}\mathcal{S}x_1 = \mathcal{S}\mathcal{A}x_1 = \mathcal{S}u_1$.

Hence $u_1 = \mathcal{B}u_1 = \mathcal{T}u_1 = \mathcal{A}u_1 = \mathcal{S}u_1$.

Uniqueness Suppose v_1 ($v_1 \neq u_1$) be other point in $\mathbb{K}$ such that

$$v_1 = \mathcal{B}v_1 = \mathcal{T}v_1 = \mathcal{A}v_1 = \mathcal{S}v_1.$$

Putting $z = u_1$ and $w = v_1, \alpha = 1$ in inequality (3.2), we obtain

$$\mathcal{F}(\mathcal{S}u_1, \mathcal{T}v_1, qt_1) = \mathcal{F}(u_1, v_1, qt_1) \geq min\begin{Bmatrix}\mathcal{F}(\mathcal{A}u_1, \mathcal{B}v_1, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{S}u_1, t_1), \\ \mathcal{F}(\mathcal{B}v_1, \mathcal{T}v_1, t_1), \mathcal{F}(\mathcal{S}u_1, \mathcal{B}v_1, t_1), \\ \mathcal{F}(\mathcal{A}u_1, \mathcal{T}v_1, t_1)\end{Bmatrix}$$

$$= min\{\mathcal{F}(u_1, v_1, t_1), \mathcal{F}(u_1, u_1, t_1)\}$$

we get $\mathcal{F}(u_1, v_1, qt_1) \geq \mathcal{F}(u_1, v_1, t_1)$.

By Lemma 2.1, we get $u_1 = v_1$.

Hence $u_1 = \mathcal{B}u_1 = \mathcal{T}u_1 = \mathcal{A}u_1 = \mathcal{S}u_1 \; and \; u_1$ is unique in $\mathbb{K}$.

Next we prove theorems for compatible mappings of type (K) and type (E) as follows:

Theorem 3.2: Let $\mathcal{A}, \mathcal{S}, \mathcal{B}$ and $\mathcal{T}$ are four self maps on a complete Menger space $(\mathbb{K}, \mathcal{F}, \Delta)$ satisfying (3.1), (3.2). Suppose that the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are reciprocally continuous and compatible of type (K). Then $u_1 = \mathcal{B}u_1 = \mathcal{T}u_1 = \mathcal{A}u_1 = \mathcal{S}u_1$, where u_1 is a unique in $\mathbb{K}$.

Proof: Now from the proof of Theorem 3.1, we can easily proved that the subsequence $\{\mathcal{S}z_{2n}\}$, $\{\mathcal{B}z_{2n+1}\}$, $\{\mathcal{T}z_{2n+1}\}$ and $\{\mathcal{A}z_{2n}\}$ of $\{w_n\}$ also converges to u_1.

Now the pairs $(\mathcal{B}, \mathcal{T})$ and $(\mathcal{A}, \mathcal{S})$ are compatible of type (K), we obtain

$$\mathcal{A}\mathcal{A}z_{2n} \to \mathcal{S}u_1, \mathcal{S}\mathcal{S}z_{2n} \to \mathcal{A}u_1 \text{ and } \mathcal{B}\mathcal{B}z_{2n} \to \mathcal{T}u_1, \mathcal{T}\mathcal{T}z_{2n+1} \to \mathcal{B}u_1 \text{ as } n \to \infty.$$

We claim that $\mathcal{B}u_1 = \mathcal{A}u_1$.

Putting $z = \mathcal{S}z_{2n}$ and $w = \mathcal{T}z_{2n+1}, \alpha = 1$ in inequality (3.2), we have

$$\mathcal{F}(\mathcal{S}\mathcal{S}z_{2n}, \mathcal{T}\mathcal{T}z_{2n+1}, qt_1) \geq min\begin{Bmatrix}\mathcal{F}(\mathcal{A}\mathcal{S}z_{2n}, \mathcal{B}\mathcal{T}z_{2n+1}, t_1), \mathcal{F}(\mathcal{A}\mathcal{S}z_{2n}, \mathcal{S}\mathcal{S}z_{2n}, t_1), \\ \mathcal{F}(\mathcal{B}\mathcal{T}z_{2n+1}, \mathcal{T}\mathcal{T}z_{2n+1}, t_1), \mathcal{F}(\mathcal{S}\mathcal{S}z_{2n}, \mathcal{B}\mathcal{T}z_{2n+1}, t_1), \\ \mathcal{F}(\mathcal{B}\mathcal{S}z_{2n}, \mathcal{T}z_{2n+1}, t_1)\end{Bmatrix}$$

Putting $n \to \infty$ and reciprocal continuity of the pairs $(\mathcal{B}, \mathcal{T})$ and $(\mathcal{A}, \mathcal{S})$, we obtain

$$\mathcal{F}(\mathcal{A}u_1, \mathcal{B}u, qt_1) \geq min\{\mathcal{F}(\mathcal{A}u_1, \mathcal{B}u_1, t_1), 1, \}$$

we get $\mathcal{F}(\mathcal{A}u_1, \mathcal{B}u_1, qt_1) \geq \mathcal{F}(\mathcal{A}u_1, \mathcal{B}u_1, t_1)$.

By Lemma 2.1, we $\mathcal{A}u_1 = \mathcal{B}u_1$.

Next we claim that $\mathcal{B}u_1 = \mathcal{S}u_1$.

Putting $z = u_1$ and $w = \mathcal{T}z_{2n+1}, \alpha = 1$ in inequality (3.2), we get

$$\mathcal{F}(\mathcal{S}u_1, \mathcal{T}\mathcal{T}z_{2n+1}, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(\mathcal{A}u_1, \mathcal{B}\mathcal{T}z_{2n+1}, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{S}u_1, t_1), \\ \mathcal{F}(\mathcal{B}\mathcal{T}z_{2n+1}, \mathcal{T}\mathcal{T}z_{2n+1}, t_1), \\ \mathcal{F}(\mathcal{S}u_1, \mathcal{B}\mathcal{T}z_{2n+1}, t_1), \\ \mathcal{F}(\mathcal{A}u_1, \mathcal{T}\mathcal{T}z_{2n+1}, t_1) \end{Bmatrix}$$

$$\mathcal{F}(\mathcal{S}u_1, \mathcal{B}u_1, qt_1) \geq min\{\mathcal{F}(\mathcal{B}u_1, \mathcal{B}u_1, t_1), \mathcal{F}(\mathcal{B}u_1, \mathcal{S}u_1, t_1)\}$$

we get $\mathcal{F}(\mathcal{S}u_1, \mathcal{B}u_1, qt_1) \geq \mathcal{F}(\mathcal{S}u_1, \mathcal{B}u_1, t_1)$.

By Lemma 2.1, we $\mathcal{S}u_1 = \mathcal{B}u_1$.

We claim that $\mathcal{S}u_1 = \mathcal{T}u_1$.

Putting $z = u_1$ and $w = u_1, \alpha = 1$ in inequality (3.2), we have

$$\mathcal{F}(\mathcal{S}u_1, \mathcal{T}u_1, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(\mathcal{A}u_1, \mathcal{B}u_1, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{S}u_1, t_1), \\ \mathcal{F}(\mathcal{B}u_1, \mathcal{T}u_1, t_1), \mathcal{F}(\mathcal{S}u_1, \mathcal{B}u_1, t_1), \\ \mathcal{F}(\mathcal{A}u_1, \mathcal{T}u_1, t_1) \end{Bmatrix}$$

$$\mathcal{F}(\mathcal{S}u_1, \mathcal{T}u_1, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(\mathcal{S}u_1, \mathcal{T}u_1, t_1), \mathcal{F}(\mathcal{B}u_1, \mathcal{B}u_1, t_1) \\ \mathcal{F}(\mathcal{A}u_1, \mathcal{A}u_1, t_1) \end{Bmatrix}$$

we get $\mathcal{F}(\mathcal{S}u_1, \mathcal{T}u_1, kt_1) \geq \mathcal{F}(\mathcal{S}u_1, \mathcal{T}u_1, t_1)$.

By Lemma 2.1, we get $\mathcal{S}u_1 = \mathcal{T}u_1$.

We claim that $u_1 = Tu_1$.

Putting $z = z_{2n}$ and $w = u_1, \alpha = 1$ in inequality (3.2), we have

$$\mathcal{F}(\mathcal{S}z_{2n}, \mathcal{T}u_1, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{B}u_1, t_1), \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{S}z_{2n}, t_1), \\ \mathcal{F}(\mathcal{B}u_1, \mathcal{T}u_1, t_1), \mathcal{F}(\mathcal{S}z_{2n}, \mathcal{B}u_1, t_1), \\ \mathcal{F}(\mathcal{B}z_{2n}, \mathcal{T}u_1, t_1) \end{Bmatrix}$$

Taking$n \to \infty$, we obtain

$$\mathcal{F}(u_1, \mathcal{T}u_1, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(u_1, \mathcal{B}u_1, t_1), \mathcal{F}(u_1, u_1, t_1) \\ \mathcal{F}(u_1, \mathcal{T}u_1, t_1) \end{Bmatrix}$$

we get $\mathcal{F}(u_1, \mathcal{T}u_1, qt_1) \geq \mathcal{F}(u_1, \mathcal{T}u_1, t_1)$.

By Lemma 2.1, we get $u_1 = \mathcal{T}u_1$.

Then$u_1 = \mathcal{B}u_1 = \mathcal{T}u_1 = \mathcal{A}u_1 = \mathcal{S}u_1$where u_1 is a unique in $\mathbb{K}$.

Uniqueness Suppose w_1 ($w_1 \neq u_1$) be other point in $\mathbb{K}$.

Putting $z = u_1$ and $w = w_1, \alpha = 1$ in inequality (3.2), we obtain

$$\mathcal{F}(\mathcal{S}u_1, \mathcal{T}w_1, qt_1) = \mathcal{F}(u_1, w_1, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(\mathcal{A}u_1, \mathcal{B}w_1, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{S}u_1, t_1), \\ \mathcal{F}(\mathcal{B}w_1, \mathcal{T}w_1, t_1), \mathcal{F}(\mathcal{S}u_1, \mathcal{B}w_1, t_1), \\ \mathcal{F}(\mathcal{A}u_1, \mathcal{T}w_1, t_1) \end{Bmatrix}$$

we get $\mathcal{F}(u_1, w_1, qt_1) \geq \mathcal{F}(u_1, w_1, t_1)$.

By Lemma 2.1, we get $u_1 = w_1$.

Hence $u_1 = \mathcal{B}u_1 = \mathcal{T}u_1 = \mathcal{A}u_1 = \mathcal{S}u_1$ *and* u_1 is unique in $\mathbb{K}$.

Theorem 3.3: Let $\mathcal{A}, \mathcal{S}, \mathcal{B}$ and $\mathcal{T}$ are four self- maps of a complete Menger space$(\mathbb{K}, \mathcal{F}, \Delta)$ satisfying (3.1), (3.2).Suppose the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible of type (E) and one of $\mathcal{S}$ and $\mathcal{A}$is continuous and one of $\mathcal{T}$ and $\mathcal{B}$ is continuous. Then$u_1 = \mathcal{B}u_1 = \mathcal{T}u_1 = \mathcal{A}u_1 = \mathcal{S}u_1$, where u_1 is a unique in $\mathbb{K}$.

Proof: From Theorem 3.1, we can easily prove the subsequence $\{\mathcal{S}z_{2n}\}$, $\{\mathcal{B}z_{2n+1}\}$,$\{\mathcal{T}z_{2n+1}\}$ and $\{\mathcal{A}z_{2n}\}$of $\{w_n\}$ also converges to u_1.

Now, suppose that one of the mappings $\mathcal{S}$ and $\mathcal{A}$ is continuous, given$\mathcal{S}$ and $\mathcal{A}$ are compatible of type (E), by Proposition 2.3, $\mathcal{A}u_1 = \mathcal{S}u_1$.

Since $\mathcal{S}(\mathbb{K}) \subset \mathcal{B}(\mathbb{K})$ and hence we can find a point $v_1 \in \mathbb{K}$satisfying$\mathcal{S}u_1 = \mathcal{B}v_1$.

We claim that $\mathcal{S}u_1 = \mathcal{T}v_1$.

On setting $z = u_1$ and $w = v_1, \alpha = 1$ in inequality (3.2), we obtain

$$\mathcal{F}(\mathcal{S}u_1, \mathcal{T}v_1, qt_1) \geq min\begin{Bmatrix}\mathcal{F}(\mathcal{A}u_1, \mathcal{B}v_1, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{S}u_1, t_1), \\ \mathcal{F}(\mathcal{B}v_1, \mathcal{T}v_1, t_1), \mathcal{F}(\mathcal{S}u_1, \mathcal{B}v_1, t_1), \\ \mathcal{F}(\mathcal{A}u_1, \mathcal{T}v_1, t_1)\end{Bmatrix}$$

$$= min\begin{Bmatrix}\mathcal{F}(\mathcal{A}u_1, \mathcal{S}u_1, t_1), \mathcal{F}(\mathcal{S}u_1, \mathcal{S}u_1, t_1) \\ , \mathcal{F}(\mathcal{S}u_1, \mathcal{B}v_1, t_1), \\ \mathcal{F}(\mathcal{S}u_1, \mathcal{T}v_1, t_1)\end{Bmatrix}$$

we get $\mathcal{F}(\mathcal{S}u_1, \mathcal{T}v_1, qt_1) \geq \mathcal{F}(\mathcal{S}u_1, \mathcal{T}v_1, t_1)$.

By Lemma 2.1, we get $\mathcal{S}u_1 = \mathcal{T}v_1$. Thus we have $\mathcal{A}u_1 = \mathcal{S}u_1 = \mathcal{T}v_1 = \mathcal{B}v_1$.

We claim that $\mathcal{S}u_1 = u_1$.

Putting $z = u_1$ and $w = z_{2n+1}, \alpha = 1$ in inequality (3.2) we have

$$\mathcal{F}(\mathcal{S}u_1, \mathcal{T}z_{2n+1}, qt_1) \geq min\begin{Bmatrix}\mathcal{F}(\mathcal{A}u_1, \mathcal{B}z_{2n+1}, t_1), \mathcal{F}(\mathcal{A}u_1, \mathcal{S}u_1, t_1), \\ \mathcal{F}(\mathcal{B}z_{2n+1}, \mathcal{T}z_{2n+1}, t_1), \mathcal{F}(\mathcal{S}u_1, \mathcal{B}z_{2n+1}, t_1), \\ \mathcal{F}(\mathcal{A}u_1, \mathcal{T}z_{2n+1}, t_1)\end{Bmatrix}$$

$$= min\{\mathcal{F}(u_1, u_1, t_1), \mathcal{F}(\mathcal{S}u_1, u_1, t_1)\}$$

we get$\mathcal{F}(\mathcal{S}u_1, u_1, qt_1) \geq \mathcal{F}(\mathcal{S}u_1, u_1, t_1)$.

Lemma 2.1gives,$u_1 = \mathcal{S}u_1$. Hence $u_1 = \mathcal{B}u_1 = \mathcal{T}u_1 = \mathcal{A}u_1 = \mathcal{S}u_1$.

Assume that $\mathcal{T}$ and $\mathcal{B}$ are compatible of type (E) and one of the mappings $\mathcal{T}$ and $\mathcal{B}$ is continuous. Then we get $\mathcal{B}v_1 = \mathcal{T}v_1 = u_1$.

By Proposition 2.3, we have $\mathcal{B}\mathcal{B}v_1 = \mathcal{B}\mathcal{T}v_1 = \mathcal{T}\mathcal{B}v_1 = \mathcal{T}\mathcal{T}v_1$.

Hence $\mathcal{B}u_1 = \mathcal{T}u_1$.

We claim that $u_1 = \mathcal{T}u_1$.

Putting $z = z_{2n}$ and $w = u_1, \alpha = 1$ in inequality (3.2), we have

$$\mathcal{F}(\mathcal{S}z_{2n}, \mathcal{T}u_1, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{B}u_1, t_1), \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{S}z_{2n}, t_1), \\ \mathcal{F}(\mathcal{B}u_1, \mathcal{T}u_1, t_1), \mathcal{F}(\mathcal{S}z_{2n}, \mathcal{B}u_1, t_1), \\ \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{T}u_1, t_1) \end{Bmatrix}$$

Letting $n \to \infty$ we obtain

$$\mathcal{F}(\mathcal{S}u_1, u_1, qt_1) \geq min\{\mathcal{F}(u_1, u_1, t_1), \mathcal{F}(\mathcal{S}u_1, u_1, t_1)\}$$

we get $\mathcal{F}(\mathcal{S}u_1, u_1, qt_1) \geq \mathcal{F}(\mathcal{S}u_1, u_1, t_1)$.

By Lemma 2.1, we $\mathcal{S}u_1 = u_1$.

Since $\mathcal{S}(\mathbb{K}) \subset \mathcal{B}(\mathbb{K})$ and hence exists a point $p_1 \in \mathbb{K}$ such that $u_1 = \mathcal{S}u_1 = \mathcal{B}p_1$.

We claim that $u_1 = \mathcal{T}p_1$.

Putting $z = \mathcal{S}z_{2n}$ and $w = p_1, \alpha = 1$ in inequality (3.2),we have

$$\mathcal{F}(\mathcal{S}\mathcal{S}z_{2n}, \mathcal{T}p_1, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(\mathcal{A}\mathcal{S}z_{2n}, \mathcal{B}p_1, t_1), \mathcal{F}(\mathcal{A}\mathcal{S}z_{2n}, \mathcal{S}z_{2n}, t_1), \\ \mathcal{F}(\mathcal{B}p_1, \mathcal{T}p_1, t_1), \\ \mathcal{F}(\mathcal{S}\mathcal{S}z_{2n}, \mathcal{B}p_1, t_1), \mathcal{F}(\mathcal{A}\mathcal{S}z_{2n}, \mathcal{T}p_1, t_1) \end{Bmatrix}$$

Taking$n \to \infty$, we obtain

$$\mathcal{F}(u_1, \mathcal{T}p_1, qt_1) \geq min\{\mathcal{F}(u_1, \mathcal{T}p_1, t_1), \mathcal{F}(u_1, u_1, t_1)\}$$

we get $\mathcal{F}(u_1, \mathcal{T}p_1, qt_1) \geq \mathcal{F}(u_1, \mathcal{T}p_1, t_1)$.

By Lemma 2.1, we get $u_1 = \mathcal{T}p_1$.Since$\mathcal{B}$ and $\mathcal{T}$ are compatible of type (R) and $\mathcal{B}p_1 = \mathcal{T}p_1 = u_1$, by Proposition 2.3, $\mathcal{B}\mathcal{T}p_1 = \mathcal{T}\mathcal{B}p_1$ and hence $\mathcal{B}u_1 = \mathcal{B}\mathcal{T}p_1 = \mathcal{T}\mathcal{B}p_1 = \mathcal{T}u_1$.

We claim that $u_1 = \mathcal{T}u_1$.

Putting $z = z_{2n}$ and $w = u_1, \alpha = 1$ in inequality (3.2), we have

$$\mathcal{F}(\mathcal{S}z_{2n}, \mathcal{T}u_1, qt_1) \geq min \begin{Bmatrix} \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{B}u_1, t_1), \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{S}z_{2n}, t_1), \\ \mathcal{F}(\mathcal{B}u_1, \mathcal{T}u_1, t_1) \\ , \mathcal{F}(\mathcal{S}z_{2n}, \mathcal{B}u_1, t_1), \mathcal{F}(\mathcal{A}z_{2n}, \mathcal{T}u_1, t_1) \end{Bmatrix}$$

Letting $n \to \infty$ we have

$$\mathcal{F}(u_1, \mathcal{T}u_1, qt_1) \geq min\{\mathcal{F}(u_1, \mathcal{T}u_1, t_1), \mathcal{F}(u_1, u_1, t_1)\}$$

we get $\mathcal{F}(u_1, \mathcal{T}u_1, qt_1) \geq \mathcal{F}(u_1, \mathcal{T}u_1, t_1)$.

By Lemma 2.1, we $\mathcal{T}u_1 = u_1$.

Then $u_1 = \mathcal{B}u_1 = \mathcal{T}u_1 = \mathcal{A}u_1 = \mathcal{S}u_1$, where u_1 is a unique in $\mathbb{K}$.

4 Conclusion

In this paper, we gave new fixed point theorems for variants of compatible mapping in menger space. We hope that out study contributes to the development of these results by other researchers.

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Competing Interests

Authors have declared that no competing interests exist.

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