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Restrained Global Defensive Alliances on Some Special Classes of Graphs

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Let $G = (V(G), E(G))$ be a graph. A set $S \subseteq V$ is a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. A *restrained dominating set* in G is a set $S \subseteq V(G)$ where every vertex in $V(G) \setminus S$ is adjacent to a vertex in S as well as another vertex in $V(G) \setminus S$. A *defensive alliance* in G is a nonempty set of vertices $S \subseteq V(G)$ if for every vertex $v \in S$, we have $|N[v] \cap S| \geq |N(v) \cap (V(G) \setminus S)|$. A defensive alliance S is called *global* if it effects every vertex in $V(G) \setminus S$, that is, every vertex in $V(G) \setminus S$ is adjacent to at least one member of the alliance S. It is known that graphs may represent different situations depending on how certain conditions were used. This study focused on those situations where restrained global defensive

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alliances were applied. Here, we investigate the formation and properties of restrained global defensive alliances within graphs, specifically focusing on graphs resembling centipede graphs, sunlet graphs, or helm graphs. We analyze how these alliances behave within these graph structures and identify key characteristics, which we label as 'characterizations.' Additionally, we determine the minimum cardinalities of these alliances, referred to as 'restrained global defensive alliance numbers,' which serve the purpose of establishing efficient networks. Through our examination, we aim to provide insights into the dynamics and efficiency of restrained global defensive alliances within these graph configurations.

Keywords: Dominating set; restrained dominating set; defensive alliance; global defensive alliance; restrained global defnsive alliance.

2010 Mathematics Subject Classification: 05C35.

1 Introduction

Alliances in graphs is one of the topics in graph theory that has been a subject of research for many years. It laid the groundwork for specific alliances and evolved in complexity as more properties were incorporated to form sets that represent real-life situations [1]. Some of these alliances include defensive, offensive, and global defensive alliances [2]. On the other hand, domination in graphs also became the basis for different types of sets known in the present [3]. One of these includes restrained domination [4].

Recently, Consistente L. F. and Cabahug I. S., introduced a new type of alliance called restrained global defensive alliances. They established some inherent properties of restrained global defensive alliances and also determined the restrained global defensive alliance number on complete graphs, complete bipartite graphs, and path graphs [5].

This study extended the restrained global defensive alliances in graphs to some graph families. It established some characterizations in centipede graphs, sunlet graphs, and helm graphs. Moreover, it also formulated the restrained global defensive alliance number on the above-mentioned graphs.

2 Preliminary Notes

Some definitions of the concepts covered in this study are included below. Here, we use V and E to indicate the vertex set $V(G)$ and edge set $E(G)$, respectively, when the graph G is understood. You may refer on the remaining terms and definitions in [6] and [7]. Moreover, note that this study is limited to finite, undirected, and simple graphs.

Definition 2.1. [8] The ***centipede graph*** $C_{n,2}$ is a graph obtained by appending a single pendant edge to each vertex of graph P_n, where P_n is the spine of $C_{n,2}$.

Example 2.1. *Fig. 1 shows the centipede graph $C_{4,2}$.*

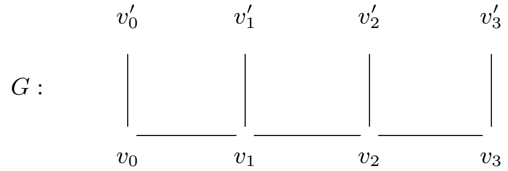


Fig. 1. Centipede graph $C_{4,2}$

Definition 2.2. [9] The ***n-sunlet graph*** S_n is the graph on $2n$ vertices obtained by attaching n pendant edges to a cycle graph C_n.

Example 2.2. *Fig. 2 shows the sunlet graph S_4.*

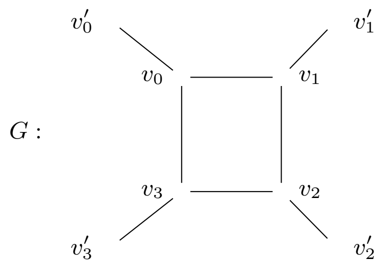


Fig. 2. Sunlet graph S_4

Definition 2.3. [9] The ***helm graph*** H_n is the graph obtained from a wheel graph W_n by adjoining a pendant edge to each node of the cycle C_n.

Example 2.3. *Fig. 3 shows the helm graph H_4.*

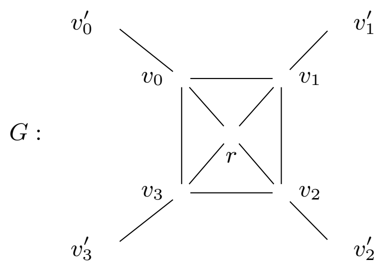


Fig. 3. Helm graph H_4

Definition 2.4. [6] A set S of vertices of $G = (V, E)$ is a ***dominating set*** if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The minimum cardinality among the dominating sets of G is called the ***domination number*** of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is referred to as a minimum dominating set.

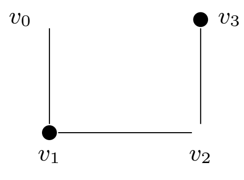


Fig. 4. Domination in path P_4

Example 2.4. *Consider the path graph P_4 in Figure 4. It can be seen that the set $S = \{v_1, v_3\}$ is a dominating set of P_4. This implies that $\gamma(P_4) \leq 2$. But any singleton subset of $V(P_4)$ is not a dominating set of P_4, meaning $\gamma(P_4) > 1$ or $\gamma(P_4) \geq 2$. Hence, we have $\gamma(P_4) = 2$.*

Definition 2.5. [4] A ***restrained dominating set*** in a graph $G = (V, E)$ is a set $S \subseteq V$ where every vertex in $V \setminus S$ is adjacent to a vertex in S as well as another vertex in $V \setminus S$. In this case, the induced subgraph $\langle V \setminus S \rangle$ has no isolated vertices. The ***restrained domination number*** of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G.

Example 2.5. *In Fig. 4, observe that the set $S_1 = \{v_1, v_3\}$ does not qualify as a restrained dominating set, as the vertex $v_0 \in V \setminus S_1$ is not connected to any other vertex in $V \setminus S_1$. On the other hand, the set $S_2 = \{v_0, v_3\}$ in Figure 5 does constitute a restrained dominating set, given that the induced subgraph of $V \setminus S_2$ contains no isolated vertices and S_2 being a dominating set in P_4.*

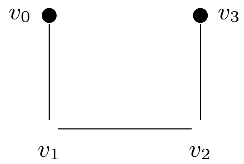


Fig. 5. Restrained domination in path P_4

Definition 2.6. [2] A ***defensive alliance*** in a graph $G = (V, E)$ is a nonempty set of vertices $S \subseteq V$ if for every vertex $v \in S$, we have $|N[v] \cap S| \geq |N(v) \cap (V \setminus S)|$. In this case, by strength of numbers, we say that every vertex in S is defended from possible attack of vertices in $V \setminus S$. A defensive alliance S is called ***global*** if it effects every vertex in $V \setminus S$, that is, every vertex in $V \setminus S$ is adjacent to at least one member of the alliance S. In this case, S is also a dominating set. The ***global defensive alliance number*** of G, denoted $\gamma_a(G)$, is the minimum size around all the global defensive alliances of G.

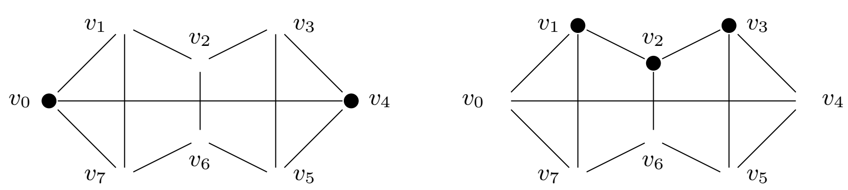
Example 2.6. *In Fig. 6, a set of vertices $S_1 = \{v_0, v_4\}$ (in Fig.6a) is an example of a defensive alliance since*

$$|N[v_0] \cap S_1| = |\{v_0, v_4\}| = 2 \geq 2 = |\{v_1, v_7\}| = |N(v_0) \cap (V \setminus S_1)|$$

and

$$|N[v_4] \cap S_1| = |\{v_0, v_4\}| = 2 \geq 2 = |\{v_3, v_5\}| = |N(v_4) \cap (V \setminus S_1)|.$$

Notice that vertices v_2 and v_6 is not adjacent to any vertex in S_1, so S_1 is not a dominating set. Hence, S_1 is not a global defensive alliance. On the other hand, by doing the same process as S_1, we can verify that set $S_2 = \{v_1, v_2, v_3\}$ (in Fig.6b) is a defensive alliance that is also a dominating set. Hence, set S_2 is a global defensive alliance.



$Fig.6a$ $Fig.6b$

Fig. 6. Defensive alliance and global defensive Alliance

Definition 2.7. [5] A ***restrained global defensive alliance*** of a graph $G = (V, E)$ is a set S of vertices of G that is restrained and global defensive. A set S with the least number of vertices is called a minimun restrained global defensive alliance. The cardinality of a minimum restrained global defensive alliance is called ***restrained global defensive alliance number*** denoted by $\gamma_{ra}(G)$.

Example 2.7. *In Fig. 7, a set $S = \{v_0, v_3, v_6, v_7, v_8, v_9\}$ is identified as a restrained dominating set in G since every vertex in $V \setminus S$ is adjacent to atleast one vertex in S and $\langle V \setminus S \rangle$ has no isolated vertices. Moreover, we have*

$$|N[v_0] \cap S| = 1 \geq 1 = |N(v_0) \cap (V \setminus S)|,$$
$$|N[v_3] \cap S| = 2 \geq 2 = |N(v_3) \cap (V \setminus S)|,$$
$$|N[v_6] \cap S| = 2 \geq 2 = |N(v_6) \cap (V \setminus S)|,$$
$$|N[v_7] \cap S| = 2 \geq 0 = |N(v_7) \cap (V \setminus S)|,$$
$$|N[v_8] \cap S| = 1 \geq 1 = |N(v_8) \cap (V \setminus S)|,$$
$$|N[v_9] \cap S| = 2 \geq 0 = |N(v_9) \cap (V \setminus S)|.$$

Hence, S is a defensive alliance in G. But, S is also dominating, so, S is a global defensive alliance in G. By Definition 2.7, S is a restrained global defensive alliance in G.

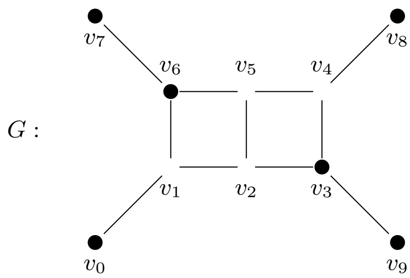


Fig. 7. Restrained global Defensive alliance

Some Known Results

The following results are taken from consistente and cabahugs study [5]. They formulate this in consideration to G being a finite, undirected and simple graph.

Theorem 2.8. *Let $G = (V, E)$ be any graph of order $n \geq 1$. Then the set V is a restrained global defensive alliance in G. As consequence, $\gamma_{ra}(G) \leq n$.*

Theorem 2.9. *Let $G = (V, E)$ be a graph with leaf vertices. If $S \subseteq V$ is a restrained global defensive alliance in G, then S contains the leaf vertices of G.*

Theorem 2.10. *Let $G = (V, E)$ be a graph with isolated vertices and $S \subseteq V$ be any restrained global defensive alliance in G. If v is an isolated vertex in G, then $v \in S$.*

Corollary 2.11. *If $P_n = (V, E)$ is a path graph of order $n \geq 2$, then*

$$\gamma_{ra}(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \ (\ mod\ 4)\,; \\ \frac{n+1}{2} & \text{if } n \equiv 1 \ (\ mod\ 4)\,; \\ \frac{n+2}{2} & \text{if } n \equiv 2 \ (\ mod\ 4)\,; \\ \frac{n+3}{2} & \text{if } n \equiv 3 \ (\ mod\ 4)\,. \end{cases}$$

3 Main Results

The following terms are used to represent distinct concepts: ds for dominating set, da for defensive alliance, rds for restrained dominating set, gda for global defensive alliance, and $rgda$ for restrained global defensive alliance.

3.1 Centipede graphs

The following are the results established in centipede graphs:

Theorem 3.1. *Let $C_{n,2} = (V, E)$ be a centipede graph of order $2n$ with $n \geq 1$ and spine P_n. Then $S \subseteq V$ is a restrained global defensive alliance if and only if the following holds:*

i. Every leaf vertices of $C_{n,2}$ are in S;

ii. $\langle V(P_n) \setminus S\rangle$ has no isolated vertices.

Proof. Let $C_{n,2} = (V, E)$ be a centipede graph of order $2n$ with $n \geq 1$ and $S \subseteq V$ be a $rgda$. Suppose that i and ii are false. Then either i or ii must not be true. Observe the following cases.

- Case 1 : i is false. Then there exist a leaf vertex of $C_{n,2}$ that is not in S. This means that Theorem 2.9 is not satisfied, a contradiction. Hence, every leaf vertices of S_n must be in S. This proves i.
- Case 2 : ii is false. Then $\langle V(P_n) \setminus S\rangle$ contains at least one isolated vertex v. Since i is true, then $v \in V \setminus S$ is not adjacent to another vertex in $v \in V \setminus S$, so S is not a rds, a contradiction. Hence, $\langle V(P_n) \setminus S\rangle$ must have no isolated vertices. This proves ii.

Conversely, let $S \subseteq V$ be a set in a centipede graph $C_{n,2}$ that satisfies i and ii. By i, S is a ds. By, i and ii, $\langle V \setminus S\rangle = \langle V(P_n) \setminus S\rangle$ has no isolated vertices. So, S is a rds. It remains to show that S is also a da. Now, for every $v \in S$ we have either of the following cases:

- Case 1 : deg $v = 1$.
  - Subcase 1 : v not adjacent to any vertex in S. Then
  $$|N[v] \cap S| = 1 \geq 1 = |N(v) \cap V \setminus S|.$$
  - Subcase 2 : v adjacent to one vertex in S. Then
  $$|N[v] \cap S| = 2 \geq 0 = |N(v) \cap V \setminus S|.$$
- Case 2 : deg $v = 2$.
  - Subcase 1 : v adjacent to one vertex in S. Then
  $$|N[v] \cap S| = 2 \geq 1 = |N(v) \cap V \setminus S|.$$

- Subcase 2 : v adjacent to two vertices in S. Then

$$|N[v] \cap S| = 3 \geq 0 = |N(v) \cap V \setminus S|.$$

- Case 3 : $\deg v = 3$.
  - Subcase 1 : v adjacent to one vertex in S. Then

$$|N[v] \cap S| = 2 \geq 2 = |N(v) \cap V \setminus S|.$$

  - Subcase 2 : v adjacent to two vertices in S. Then

$$|N[v] \cap S| = 3 \geq 1 = |N(v) \cap V \setminus S|.$$

  - Subcase 3 : v adjacent to three vertices in S. Then

$$|N[v] \cap S| = 4 \geq 0 = |N(v) \cap V \setminus S|.$$

Since all the cases holds, S is also a da. This means that S is also a gda. Therefore, S is a $rgda$ in $C_{n,2}$. □

Lemma 3.2. *Let $C_{n,2} = (V, E)$ be a centipede graph of order $2n$ where $n \geq 1$ and $V = \{v_0, v_1, ..., v_{n-1}, v'_0, v'_1, ..., v'_{n-1}\}$ such that $v_i v_{i+1} \in E$ with $0 \leq i \leq n-2$. If v'_i are the leaf vertices of $C_{n,2}$ and $v_i v'_i \in E$ with $0 \leq i \leq n-1$, then $S = \{v'_0, v'_1, ..., v'_{n-1}\}$ is a restrained global defensive alliance in $C_{n,2}$.*

Proof. Let set $S = \{v'_0, v'_1, ..., v'_{n-1}\}$. This means that S contains all the leaf vertices of $C_{n,2}$. Moreover, $\langle V \setminus S\rangle = \langle V(P_n) \setminus S\rangle = P_n$, then $\langle V(P_n) \setminus S\rangle$ has no isolated vertices. With these, S satisfies all the conditions in Theorem 3.1. Therefore, S is a $rgda$ in $C_{n,2}$. □

Corollary 3.3. *Let $C_{n,2} = (V, E)$ be a centipede graph of order $2n$, then*

$$\gamma_{ra}(C_{n,2}) = \begin{cases} 2 & \text{if } n = 1; \\ \\ n & \text{if otherwise.} \end{cases}$$

Proof. Let $C_{n,2} = (V, E)$ be a centipede graph of order $2n$ where $n \geq 1$ and $V = \{v_0, v_1, ..., v_{n-1}, v'_0, v'_1, ..., v'_{n-1}\}$ such that $v_i v_{i+1} \in E$ with $0 \leq i \leq n-2$. Moreover $v_i v'_i \in E$ with $0 \leq i \leq n-1$ and v'_i are the leaf vertices of $C_{n,2}$. Observe the following cases.

- Case 1 : $n = 1$. This means that $C_{1,2} = P_2$. By Corollary 2.11,

$$\begin{aligned} \gamma_{ra}(C_{1,2}) &= \gamma_{ra}(P_2) \\ &= \frac{2+2}{2} \\ &= 2. \end{aligned}$$

- Case 2 : $n \geq 2$. Consider a set $S = \{v'_0, v'_1, ..., v'_{n-1}\}$. By Lemma 3.2, S is a $rgda$ in $C_{n,2}$. Now, assume that $W \subset S$. So, there exist a vertex in S that is not in W. This means that W does not satisfy Theorem 3.1(i). So, W is not a $rgda$. Hence, S is the minimum $rgda$ of S_n. Therefore,

$$\begin{aligned} \gamma_{ra}(C_{1,2}) &= |S| \\ &= |\{v'_0, v'_1, ..., v'_{n-1}\}| \\ &= |\{v_0, v_1, ..., v_{n-1}\}| \\ &= |V(P_n)| \\ &= n. \end{aligned}$$

□

Example 3.4. *Fig. 8 is a centipede graph $C_{4,2}$. Notice that the shaded vertices satisfies all the conditions in Theorem* 3.1. *Hence, this set is a rgda in $C_{4,2}$. Moreover, by Corollary* 3.3, $\gamma_{ra}(C_{4,2}) = 4$. *Hence, the shaded vertices is the minimum rgda in $C_{4,2}$.*

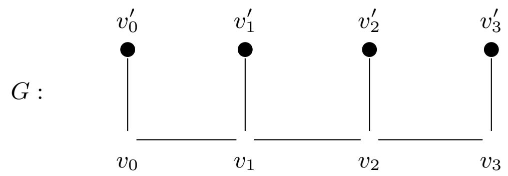


Fig. 8. ***Rgda* in centipede graph $C_{4,2}$**

3.2 Sunlet graphs

This section presents the results established in sunlet graphs:

Theorem 3.5. *Let $S_n = (V, E)$ be a sunlet graph of order $2n$ where $n \geq 3$ and C_n be the only cycle graph that can be induced by vertices of S_n. Then $S \subseteq V$ is a restrained global defensive alliance if and only if the following holds:*

i. Every leaf vertices of S_n are in S;

ii. $\langle V(C_n) \setminus S\rangle$ has no isolated vertices.

Proof. Let S be a *rgda* in a sunlet graph $S_n = (V, E)$ with $n \geq 3$ that satisfies i and ii. Suppose that i and ii are false. Then either i or ii is not true. Then we have the following cases.

- Case 1 : i is false. Then there exist a leaf vertex of S_n that is not in S. This means that Theorem 2.9 is not satisfied, a contradiction. Hence, every leaf vertices of S_n must be in S. This proves i.
- Case 2 : ii is false. Then $\langle V(C_n)\setminus S\rangle$ contains at least one isolated vertex v. Since i is true, then $v \in V \setminus S$ is not adjacent to another vertex in $v \in V \setminus S$, so S is not *rgda*, contradiction. Hence, $\langle V(C_n) \setminus S\rangle$ must have no isolated vertices. This proves ii.

Conversely, let $S \subseteq V$ be a set in a sunlet graph S_n with $n \geq 3$ that satisfies i and ii. By i, S is nonempty. Moreover, since every vertex in $V(Cn) \subset V$ is adjacent to a unique leaf vertex, S is a *ds*. By ii, $\langle V \setminus S\rangle = \langle V(C_n) \setminus S\rangle$ has no isolated vertices, so S is a *rds*. In addition, knowing that every vertex in S_n is adjacent to atmost three vertices, for every $a \in S$ we have

- Case 1 : a is a leaf vertex

Subcase 1 : a is adjacent to a vertex in S. Then

$$|N[a] \cap S| = 2 \geq 0 = |N(a) \cap V \setminus S|.$$

Subcase 2 : a is not adjacent to a vertex in S. Then

$$|N[a] \cap S| = 1 \geq 1 = |N(a) \cap V \setminus S|.$$

- Case 2 : a is not a leaf vertex

Subcase 1 : a is adjacent to one vertex in S. Then

$$|N[a] \cap S| = 2 \geq 2 = |N(a) \cap V \setminus S|.$$

Subcase 2 : a is adjacent to two vertex in S. Then

$$|N[a] \cap S| = 3 \geq 1 = |N(a) \cap V \setminus S|.$$

Subcase 3 : a is adjacent to three vertex in S. Then

$$|N[a] \cap S| = 4 \geq 0 = |N(a) \cap V \setminus S|.$$

Hence, S is a *da*. Therefore, S is a *rgda* in S_n. □

Lemma 3.6. *Let $S_n = (V, E)$ be a sunlet graph of order $2n$ where $n \geq 3$ and $V = \{v_0, v_1, ..., v_{n-1}, v'_0, v'_1, ..., v'_{n-1}\}$ such that $v_iv_{i+1}, v_0v_{n-1} \in E$ with $0 \leq i \leq n-2$. If v'_i are the leaf vertices of S_n and $v_iv'_i \in E$ with $0 \leq i \leq n-1$, then $S = \{v'_0, v'_1, ..., v'_{n-1}\}$ is a restrained global defensive alliance in S_n.*

Proof. Let $S = \{v'_0, v'_1, ..., v'_{n-1}\}$. This means that S contains all the leaf vertices of S_n. Moreover, $\langle V \setminus S\rangle = \langle V(C_n) \setminus S\rangle = C_n$, then $\langle V(C_n) \setminus S\rangle$ has no isolated vertices. With these, S satisfies all the conditions in Theorem 3.5. Therefore, S is a *rgda* in S_n. □

Corollary 3.7. *If S_n is a sunlet graph of order $2n$, $n \geq 3$, then*

$$\gamma_{ra}(S_n) = |V(C_n)| = n.$$

Proof. Let $S_n = (V, E)$ be a sunlet graph of order $2n$, $n \geq 3$, and $V = \{v_0, v_1, ..., v_{n-1}, v'_0, v'_1, ..., v'_{n-1}\}$ such that $v_iv_{i+1}, v_0v_{n-1} \in E$ with $0 \leq i \leq n-1$. Moreover, $v_iv'_i \in E$ such that $0 \leq i \leq n-1$ where v'_i are the leaf vertices in S_n.

Consider the set $S = \{v'_0, v'_1, ..., v'_{n-1}\}$. By Lemma 3.6, S is a *rgda*. Assume that $W \subset S$. So, there exist vertices in S that is not in W. This means that W does not satisfy Theorem 3.5(i). So, W is not a *rgda*. Hence, S is the minimum *rgda* of S_n. Therefore,

$$\begin{aligned}
\gamma_{ra}(S_n) &= |S| \\
&= |\{v'_0, v'_1, ..., v'_{n-1}\}| \\
&= |\{v_0, v_1, ..., v_{n-1}\}| \\
&= |V(C_n)| \\
&= n.
\end{aligned}$$

□

Example 3.8. *Fig. 9 is a sunlet graph S_4. Notice that the shaded vertices are the collection of all leaf vertices of S_4 and it satisfies all the conditions in Theorem 3.5. Hence, this set is a rgda in S_4. Moreover, by Corollary 3.7, $\gamma_{ra}(S_4) = 4$. Hence, the shaded vertices is the minimum rgda in S_4.*

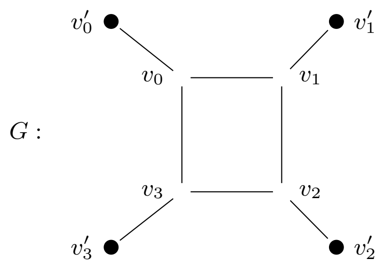


Fig. 9. ***Rgda* in sunlet graph S_4**

Helm graphs

The following are the results established in helm graphs:

Theorem 3.9. *Let $H_n = (V, E)$ be a helm graph of order $2n + 1$, $n \geq 3$, where r is the root vertex of W_n, and C_n be the largest cycle induced by the vertices of H_n. Then $S \subseteq V$ is a restrained global defensive alliance if and only if the following holds:*

i. *Every leaf vertices of H_n are in S;*

ii. $\langle V(W_n) \smallsetminus S\rangle$ *has no isolated vertices;*

iii. $(V(W_n) \cap S) \neq \emptyset$;

iv. $\langle V(W_n) \cap S\rangle$ *has no isolated vertices;*

v. $|V(C_n) \cap S| \geq \left\lfloor \frac{|V(C_n)|}{2} \right\rfloor$ *if $r \in S$.*

Proof. Let S be a $rgda$ in a helm graph $H_n = (V, E)$ of order $2n + 1$. Let $W_n \subset S$ be vertices such that $\langle W_n \rangle$ is a wheel graph of order $n + 1$ with r being its root vertex.

Suppose that i, ii, iii, iv, and v are not true. Then either i, ii, iii, iv, or iv is false.

- Case 1 : i is false. Then there exist leaf vertices that is not in S. This is not possible since by Theorem 2.9, every leaf vertices of H_n must be in S. This proves i.
- Case 2 : ii is false. Then $\langle V(W_n) \smallsetminus S\rangle$ contains an isolated vertex. Since i is true, $\langle V \smallsetminus S\rangle = \langle V(W_n) \smallsetminus S\rangle$ has isolated vertices. So, S is not a rds, a contradiction. Hence, $\langle V(W_n) \smallsetminus S\rangle$ must have no isolated vertices. This proves ii.
- Case 3 : iii is false. Then $(V(W_n) \cap S) = \emptyset$. So, r is not adjacent to another vertex in S. This means that S is not a ds, a contradiction. Hence, $(V(W_n) \cap S) \neq \emptyset$. This proves iii.
- Case 4 : iv is false. Then $\langle V(W_n) \cap S\rangle$ contains at least one isolated vertex v. Since i, ii, and iii are true, for every $v \in V(W_n) \cap S$ we have

Subcase 1 : $v \in V(W_n) \smallsetminus \{r\}$. Then $|N[v] \cap S| = 2 \ngeq 3 = |N(v) \cap V \smallsetminus S|$, so, S is not a da, a contradiction.

Subcase 2 : $v = r$. Then $|N[v] \cap S| = 1 \ngeq n = |N(v) \cap V \smallsetminus S|$, so, S is not a da, a contradiction. Hence, $\langle V(W_n) \cap S\rangle$ has no isolated vertices. This proves iv.

- Case 5 : v is false. Then $r \in S$ and $|V(C_n) \cap S| \ngeq \left\lfloor \frac{|V(C_n)|}{2} \right\rfloor$. This means that

$$\begin{aligned} |N[r] \cap S| &= |V(C_n) \cap S| + |\{r\}| \\ &\ngeq \left\lfloor \frac{|V(C_n)|}{2} \right\rfloor + 1 \\ &\leq |V(C_n)| - \left\lfloor \frac{|V(C_n)|}{2} \right\rfloor \\ &\leq |N(r) \cap V \smallsetminus S|. \end{aligned}$$

  So, S is not a da, a contradiction. Hence, $|V(C_n) \cap S| \geq \left\lfloor \frac{|V(C_n)|}{2} \right\rfloor$ if $r \in S$. This proves v.

Conversely, let $S \subseteq V$ be set in a helm graph H_n with $n \geq 3$ that satisfies i, ii, iii, iv, and v. By i, S is nonempty and every vertex in $V(W_n) \smallsetminus \{r\} = C_n$ is adjacent to a unique leaf vertex. It remains to show that r is adjacent to a vertex in S. We know that r is adjacent to any vertex in C_n, so by iii, S is a ds. By ii, $\langle V \smallsetminus S\rangle = \langle V(W_n) \smallsetminus S\rangle$ has no isolated vertices, so, S is a rds. By i, ii, iii, iv and v, for every $v \in S$ we have

- Case $1:$ v is a leaf vertex

Subcase $1:$ v is not adjacent to any vertex in S. Then

$$|N[v] \cap S| = 1 \geq 0 = |N(v) \cap V \setminus S|.$$

Subcase $2:$ v is adjacent to one vertex in S. Then

$$|N[v] \cap S| = 2 \geq 0 = |N(v) \cap V \setminus S|.$$

- Case $2:$ $v \in C_n \cap S$.

Subcase $1:$ v is adjacent to two vertices in S. Then

$$|N[v] \cap S| = 3 \geq 2 = |N(v) \cap V \setminus S|.$$

Subcase $2:$ v is adjacent to three vertices in S. Then

$$|N[v] \cap S| = 4 \geq 1 = |N(v) \cap V \setminus S|.$$

Subcase $3:$ v is adjacent to four vertices in S. Then

$$|N[v] \cap S| = 5 \geq 0 = |N(v) \cap V \setminus S|.$$

- Case $3:$ $v = r$. Then

$$|N[v] \cap S| = |V(C_n) \cap S| + |\{v\}| \geq \left\lfloor \frac{|V(C_n)|}{2} \right\rfloor + 1 \geq |V(C_n) \setminus S| = |N(v) \cap V \setminus S|.$$

Hence, S is a da. Therefore, by i, ii, iii, iv, and v, S is a $rgda$ in H_n. □

Lemma 3.10. *Let $H_n = (V, E)$ be a helm graph of order $2n + 1$ where $n \geq 3$ and $V = \{v_0, v_1, ..., v_{n-1}, v'_0, v'_1, ..., v'_{n-1}, r\}$ such that $v_i v_{i+1}, v_0 v_{n-1} \in E$ with $0 \leq i \leq n-2$. If v'_i are the leaf vertices of H_n and $v_i v'_i, r v_i \in E$ with $0 \leq i \leq n-1$, then $S = \{v'_0, v'_1, ..., v'_{n-1}, v_0, v_1\}$ is a restrained global defensive alliance in H_n.*

Proof. Let $S = \{v'_0, v'_1, ..., v'_{n-1}, v_0, v_1\}$. Notice that all the leaf vertices $v'_0, v'_1, ..., v'_{n-1}$ of H_n are in S. Moreover, $\langle V(W_n) \setminus S\rangle = \{r, v_2, v_3, ..., v_{n-1}\}$ and $v_2, v_3, ..., v_{n-1}$ are adjacent to the root vertex r. So, $\langle V(W_n) \setminus S\rangle$ has no isolated vertices. In addition, $V(W_n) \cap S = \{v_0, v_1\}$. This means that $|V(W_n) \cap S| \neq \emptyset$. On the other hand, $V(W_n) \cap S = \{v_0, v_1\}$ and $v_0 v_1 \in E$. So, $\langle V(W_n) \cap S\rangle$ has no isolated vertices. With all these, S satisfies all the conditions in Theorem 3.9. Therefore, $S \subseteq V$ is a $rgda$ in H_n. □

Corollary 3.11. *Let $H_n = (V, E)$ be a helm graph of order $2n + 1$ where $n \geq 3$, then $\gamma_{ra}(H_n) = n + 2$.*

Proof. Let $H_n = (V, E)$ be a helm graph of order $2n + 1$ where $n \geq 3$ and vertex set $V = \{v_0, v_1, ..., v_{n-1}, v'_0, v'_1, ..., v'_{n-1}, r\}$ such that $v_i v_{i+1}, v_0 v_{n-1} \in E$ with $0 \leq i \leq n-2$. Moreover, $v_i v'_i, r v_i \in E$ such that $0 \leq i \leq n-1$ and v'_i are the leaf vertices of H_n.

Consider a set $S = \{v'_0, v'_1, ..., v'_{n-1}, v_0, v_1\}$. By Lemma 3.10, S is a $rgda$. Assume that $W \subset S$. So, there exist vertices in S that is not in W. This leads to the following cases.

- Case $1:$ W does not contain either $v'_0, v'_1, ..., v'_{n-1}$. Then W is not a $rgda$ since Theorem 3.9(i) is not satisfied.
- Case $2:$ W does not contain either but not both v_0 or v_1. Then W is not a $rgda$ since Theorem 3.9(iv) is not satisfied
- Case $3:$ W does not contain both v_0 and v_1. Then W is not a $rgda$ since Theorem 3.9(iii) is not satisfied.

Knowing that neither of these cases holds, S is a minimum *rgda* in H_n. Therefore,
$\gamma_{ra}(H_n) = |S| = |\{v'_0, v'_1, ..., v'_{n-1}, v_0, v_1\}| = n + 2.$ □

Example 3.12. *Fig. 10 is a helm graph H_4. Notice that the shaded vertices satisfies all the conditions in Theorem* 3.9. *Hence, this set is a rgda in H_4. Moreover, by Corollary* 3.11, $\gamma_{ra}(H_4) = 4 + 2 = 6$. *Therefore, the set containing the shaded vertices is a minimum rgda in H_4.*

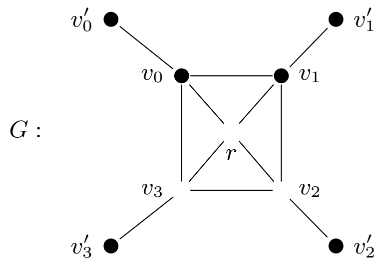


Fig. 10. ***Rgda* in helm graph H_4**

4 Conclusion

In this article, restrained global defensive alliances in centipede graphs, sunlet graphs, and helm graphs are studied. Furthermore, the restrained global defensive alliance number is also determined. Lastly, we intend to examine the restrained global defensive alliances and restrained global defensive alliance number for many unstudied graph families in the future.

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Competing Interests

The authors declare that they have no competing interests.

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