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Proving Maximal Linear Loose Tangle as a Linear Tangle

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Short Research Article

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Abstract

Linear-width is a well-regarded width parameter in graph theory. The constructs of linear loose tangle and linear tangle present obstacles to attaining linear-width. In this succinct paper, our primary focus will be the exploration of maximal linear loose tangles.

Keywords: Maximal linear loose tangle; linear tangle; linear loose tangle; linear tangle matroid.

1 Introduction

The study of graph width parameters is pervasive across various fields, including matroid theory, lattice theory, computer science, game theory, network theory, artificial intelligence, graph theory, and discrete mathematics, as evidenced by a substantial body of literature (see [1-22,23-35] for references). These graph width parameters, often explored in relation to obstruction, have spawned a considerable amount of scholarly research.

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One such graph width parameter is linear-width, which has also been extensively studied (see [1,3,6,11] for references). Linear tangle, initially introduced in [1], plays a crucial role in determining whether a linear width is at most k, where $k+1$ represents the order of the tangle. Likewise, linear loose tangle, which was first introduced in [4], also serves as an obstruction to linear-width at most k if its order is $k+1$.

In this concise paper, we focus on maximal linear loose tangle. While it may lack novelty, our objective is to contribute to the research on graph width parameters.

2 Definitions and Notations

In this section, we present the mathematical definitions and notations for each concept.

2.1 Symmetric submodular function

The definition of a symmetric submodular function is given below.

Definition 1: Let X be a finite set. A function $f: X \rightarrow \mathbb{N}$ is called symmetric submodular if it satisfies the following conditions:

· $\forall A \subseteq X, f(A) = f(X \setminus A)$.
· $\forall A, B \subseteq X, f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$.

In this short paper, a pair (X, f) of a finite set X and a symmetric submodular function f is called a connectivity system.

It is known that a symmetric submodular function f satisfies the following properties:

Lemma 1 [18]: A symmetric submodular function f satisfies:

1. $\forall A \subseteq X, f(A) \geq f(\emptyset) = f(X)$.
2. $\forall A, B \subseteq X, f(A) + f(B) \geq f(A \setminus B) + f(B \setminus A)$.

In this paper, we use the notation f for a symmetric submodular function, a finite set X, and a natural number k.
A set A is k-efficient if $f(A) \leq k$.

2.2 Linear tangle

The definition of a linear tangle on a connectivity system (X,f) is provided below. Please note that in reference [1], the order is denoted as k instead of $k + 1$. It is important to note that when a graph does not contain any pendant vertices, the equivalence still holds.

Definition 2 [1]: Let X represent a finite set and f denote a symmetric submodular function delineated over X, mapping to the non-negative integers. A linear tangle of order $k+1$ on a connectivity system (X,f) is a family L of k-efficient subsets of X, satisfying the following axioms:

(L1) $\emptyset \in L$,
(L2) For each k-efficient subset A of X, exactly one of A or $X \setminus A$ in L,
(L3) If $A,B \in L$, $e \in X$, and $f(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq X$ holds.

A linear tangle possesses the following characteristics:

Lemma 2. Let X represent a finite set and f denote a symmetric submodular function delineated over X, mapping to the non-negative integers. A linear tangle L of order $k+1$ on a connectivity system (X,f) satisfies following axioms:

(L4)If $e \in X$, then $X \setminus \{e\} \notin L$

Proof. Assume, towards a contradiction, that there exists an element e in set X such that $X\setminus \{e\}$ belongs to L. Under this assumption, by the properties of symmetric submodular functions, we have $f(X/\{e\}) = f(\{e\}) \leq k$. However, this condition conflicts with axiom (L3), leading us to a contradiction. This completes the proof.

2.3 Linear loose tangle

The definition of a linear loose tangle on a connectivity system (X,f) is given below.

Definition 3[4]: Let X represent a finite set and f denote a symmetric submodular function delineated over X, mapping to the non-negative integers. Let T be a subset of 2^X that represents the order $k+1$ of linear loose tangle, and satisfies the following conditions:

(IN) $\forall e \in X$, if $f(\{e\}) \leq k$, then $\{e\} \in T$.
(LTSU) For any $A \in T$ and any element $e \in X$, if $B \subseteq A \cup \{e\}$ and $f(B) \leq k$ and $f(\{e\}) \leq k$, then $B \in T$.
(IW) $X \notin T$.

A maximal linear loose tangle is one that cannot be extended further without violating its defining conditions.

3 Result of this Paper: Proving Maximal Linear Loose Tangle as a Linear Tangle

We consider the case when the linear loose tangle on a connectivity system (X,f) is maximized. The lemmas\theorems and their corresponding proofs are presented below. Notably, Lemma 3 uses a proof that relies on the concept of maximality.

Lemma 3. Let X represent a finite set and f denote a symmetric submodular function delineated over X, mapping to the non-negative integers. If T is a Maximal linear loose tangle of order $k+1$ on (X,f), T saitisfies axiom (L2).

Proof of Lemma 3: Assume that T is a Maximal linear loose tangle of order $k+1$ on (X,f). By definition, T is a subset of 2^X that represents order $k+1$ of a linear loose tangle, and satisfies conditions (IN), (LTSU), and (IW).

Consider a k-efficient subset A of X, that is $f(A) \leq k$. We aim to prove that exactly one of A or $X\setminus A$ is in T. By axiom (LTSU), since A is k-efficient, A should be in T. For $X\setminus A$ to also be in T, there must exist an element e in $X\setminus A$ such that $f(\{e\}) \leq k$. However, because $A \cup X\setminus A = X$, and we know from axiom (IW) that $X \notin T$, we have a contradiction. In other words, the logical connection is that condition (LTSU) potentially allows us to include both A and $X\setminus A$ in T, but doing so would cause us to violate condition (IW), and hence we conclude that exactly one of A and $X\setminus A$ is in T.

Next, suppose that neither A nor $X\setminus A$ is in T. As T is a maximal linear loose tangle, we can add either A or $X\setminus A$ to T without violating any of the conditions (IN), (LTSU), and (IW). Therefore, by maximality, at least one of A or $X\setminus A$ must be in T.

Hence, exactly one of A or $X\setminus A$ is in T, proving that T satisfies axiom (L2). This concludes the proof.

Lemma 4. Let X represent a finite set and f denote a symmetric submodular function delineated over X, mapping to the non-negative integers. If T is a Maximal linear loose tangle of order $k+1$ on (X,f), T saitisfies axiom (L3).

Proof of Lemma 4: Assume that T is a maximal linear loose tangle of order $k+1$ on (X, f), meaning that T satisfies conditions (IN), (LTSU), and (IW). We need to show that for any $A, B \in T$ and $e \in X$ with $f(\{e\}) \leq k$, we have $A \cup B \cup \{e\} \neq X$.

For the sake of contradiction, assume that there exist $A, B \in T$ and $e \in X$ such that $f(\{e\}) \leq k$ and $A \cup B \cup \{e\} = X$.

Now, let's consider a set $D = A \cup B$. If $f(D) > k$, D cannot be in T by definition as it is not k-efficient. If $f(D) \leq k$, we need to consider the (LTSU) axiom carefully. If $D \in T$, then because $\{e\} \in T$ (due to $f(\{e\}) \leq k$ and axiom (IN)), we would have $D \cup \{e\} \in T$ by axiom (LTSU), as $D \cup \{e\}$ is a subset of $D \cup \{e\}$, and $f(D) \leq k$ and $f(\{e\}) \leq k$. But this would imply $X \in T$, contradicting axiom (IW) that states $X \notin T$. Hence, we conclude that our assumption that $A \cup B \cup \{e\} = X$ must be false.This concludes the proof.

Lemma 5. Let X represent a finite set and f denote a symmetric submodular function delineated over X, mapping to the non-negative integers. If T is a Maximal linear loose tangle of order $k+1$ on (X,f), T saitisfies axiom (L1).

Proof of Lemma 5: Assume that T is a maximal linear loose tangle of order k+1 on (X, f), meaning that T satisfies conditions (IN), (LTSU), and (IW). By Lemma 1, for any $A \subseteq X$, $f(A) \geq f(\emptyset)$. Since $\emptyset$ is a subset of every set, $f(\emptyset) \leq k$ holds, implying that $\emptyset$ is k-efficient.

Now, condition (IN) of T states that for every $e \in X$, if $f(\{e\}) \leq k$, then $\{e\}$ is in T. Specifically, for the empty set, which we've shown is k-efficient, this implies that $\emptyset$ should be in T. Hence, T satisfies axiom (L1), and this concludes the proof.

From Lemmas 3, 4, and 5, we can derive the following theorem:

Theorem 6. Let X represent a finite set and f denote a symmetric submodular function delineated over X, mapping to the non-negative integers. If T is a Maximal linear loose tangle of order $k+1$ on (X,f), T is linear tangle of order $k+1$ on (X,f).

4 Future Tasks: Linear Tangle Matroids

In the future, I am contemplating researching the concept of "Linear Tangle Matroids." The concept of tangle matroid was defined within the framework of matroids (see [18,36-41]). Here are definitions of tangle and tangle matroids. Please note that the connectivity function of a matroid M, denoted as$\lambda_M(A)$, for a subset A of $E(M)$, is defined as $\lambda_M(A) = r(A) + r(E(M)-A) - r(M)$.

Definition 4[23]: Let M be a matroid, and T a collection of subsets of $E(M)$. Then T is a tangle of order $k+1$ of M if

(TM1) for all $A \in T$, $\lambda_M(A) \leq k$;
(TM2) for all $A \subseteq E(M)$ with $\lambda_M(A) \leq k$, either $A \in T$ or $E(M) - A \in T$;
(TM3) if $A, B, C \in T$, then $A \cup B \cup C \neq E(M)$;
(TM4) for each $e \in E(M)$, $E(M) - \{e\} \notin T$

We plan to investigate what characteristics emerge when the axiom (TM3) is modified to the following form, (LTM3):

(LTM3) if $A, B \in T$, $e \in E(M)$, $\lambda_M(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq E(M)$.

A tangle that satisfies (LTM3) rather than (TM3) is referred to as a Linear Tangle of order $k+1$ of M.

5 Conclusion

Drawing inspiration from the ideas presented in references [24], we will contemplate a matroid M and a Linear tangle T of M of order $k+1$. Define a function $\rho: 2^{E(M)} \to N$ as follows: $\rho(A) := min\{\lambda_M(B)$: A is a subset of B and B is in $T\}$ if there exists a B such that A is a subset of B and B is in T. Otherwise, $\rho(A) := k + 1$. It has been shown in reference [18] that ρ is the rank function of a matroid on $E(M)$. Note that since it holds true in the case of a Tangle, it naturally extends to be applicable in the case of a Linear Tangle as well. If T is a Linear-tangle of a matroid M, we refer to $M(T)$ as a "Linear tangle matroid" of M. Going forward, We aim to carry out the characterization of the aforementioned Linear Tangle Matroids.

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Competing Interests

Author has declared that no competing interests exist.

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