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Local Orthonormal Basis of the Fibres of Quasi-Hemi-Slant Riemannian Submersion

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Review Article

Abstract

In this paper we recall some different Riemannian Submersions such as; Semi-Invariant, Semi-Slant, Hemi-Slant and Quasi-Hemi-Slant Submersion from Almost Hermitian manifold to Riemannian manifold. Our goal in this work is to introduce local orthonormal bases for the fibers of Quasi-Hemi-Slant Riemannian submersion which generalizes Hemi-Slant, Semi-Slant and Semi-Invariant submersion from Almost Hermitian manifold to Riemannian manifold.

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1 Introduction

The importance of Immersions and Submersions in geometry and other parts of differential geometry cannot be over emphasized. O'Neil and Gray first introduced Riemannian Submersion. See [1] [2]. The topic became an important area of study by other geometers. See [[3], [4]]. Riemannian Submersion between Riemannian manifold equipped with some additional structures of almost complex type was first of all studied by Watson. He defined Almost Hermitian submersion between Almost Hermitian manifold and proved that the base manifold and all fibers have similar kind of structure as the total space.

Many other Geometers have introduced several other kinds of submersions base on conditions associated with the submersion. Some of these well known submersions are; Invariant submersion [5], Semi-Invariant submersion [6], Anti-invariant Submersion [7], Submersion of Semi-Invariant Submanifolds of Trans-Sasakian Manifold [8], Geometry of slant Submanifolds [9], slant submersion from Almost Hermitian manifold [10], Semi-Slant submersion [11], quaternionic submersion [12], Riemannian Submersion from Almost contact metric manifolds [13], almost h-slant submersion and h-slant submersion [14], Hemi-Slant submersion [15] and the one introduced; Quasi-Hemi-Slant Riemannian Submersion [16].

Riemannian Submersion plays a very important roll in some topics in physics, It has applications in Kaluza Klein theory ([17], [18]), Yang Mills theory ([19], [20]), Supergravity and superstring theories. See ([21], [22]) for details.

A new class of Riemannian Submersion called Quasi-Hemi-Slant Riemannian Submersion was introduced.See [16]. This class of submersion generalizes Semi-Invariant, Hemi-Slant and Semi-Slant submersions defined on Almost Hermitian manifolds. This work introduces local Orthonormal Basis of the fibers of Quasi-Hemi-Slant submersion defined on Almost Hermitian manifolds.

The work is organized as follows; Section 2, we give some of the basic definitions and some important results of some Riemannian Submersion. Section 3, we recall the new Riemannian Submersion by Longwap at el [16] and introduce orthonormal bases for the fibers. Finally, in section 4 we relate the geometry of the fibre to the fundamental tensors.

2 Prelimineries

In this section we give some basic definition of Almost Hermitian Manifolds, and recall the necessary background and definition of Riemannian Submersion from an Almost Hermitian manifold to a Riemannian manifold.

Suppose that (M, g) is an $n-$dimensional Riemannian manifold with a tensor field J of type $(1, 1)$. If for every, $X, Y \in \Gamma(TM)$

$$g(JX, JY) = g(X, Y), \quad J^2 = -I,$$

then (M, g) is called an Almost Hermitian manifold, J is an almost complex structure in (M, g).

If the structure J and the connection $\bar{\nabla}$ satisfy the equation $(\bar{\nabla}_X J)Y = 0$ for all $X, Y \in \Gamma(TM)$, then M is called a Kaehler manifold. $\bar{\nabla}$ is the Levi-Civita connection on M. We define the covariant derivative of the complex structure J as;

$$\bar{\nabla}_X J)Y = \bar{\nabla}_X JY - J\bar{\nabla}_X Y.$$

Thus, if M is a Kaehler manifold then $\bar{\nabla}_X JY = J\bar{\nabla}_X Y$.See [4] for details.

Definition 2.1. [3] Suppose that (M, g) is $m-$dimensional Riemannian manifold, and (N, g) is $n-$dimensional Riemannian manifold, such that $dim(M) > dim(N)$. A surjective map $\rho : (M, g) \rightarrow (N, g)$ is a Riemannian submersion if the following conditions hold.

1. the map ρ has maximal rank,
2. if ρ_* is restricted to $(ker\rho_*)^{\perp}$, then it is linear isometry map.

Here we have that for any $y \in N$, $\rho^{-1}(y)$ is a $k-$dimensional Riemannian submanifold of M. It is called the fiber, and that $dim(M) - dim(N) = k$. Any vector field on M that is tangent to the fibers is a vertical vector, if the vector is orthogonal to the fibers then it is a horizontal vector.

Definition 2.2. *[5, p. 84] If ρ is a Riemannian Submersion from an Almost Hermitian manifold (M, g^M, J^M) to a Riemannian manifold (N, g^N). Then ρ is said to be Invariant Riemannian Submersion if the vertical distribution is invariant with respect to the complex structure J^M. That is,*

$$J^M \ker \rho_* = \ker \rho_*.$$

This follows that;

Theorem 2.1. *[5] If $\rho : (M, g^M, J_M) \rightarrow (N, g^N, J^N)$ is a Riemannian Submersion from Almost Hermitian manifold onto an Almost Hermitian manifold. Then the horizontal distribution is invariant with respect to J_M.*

Almost Hermitian submersion that exists between Almost Hermitian manifolds was define by Watson. He showed that the base manifold and the fibers of such submersion have similar structures with the total space.[4].

Definition 2.3. *[7] Suppose that M is an Almost Hermitian manifold with a hermitian metric g^M and almost complex structure J. Let N be a Riemannian manifold with a Riemannian metric g^N. If there is a Riemannian Submersion $\rho : M \longrightarrow N$ with the property that, $J \ker \rho_* \subseteq (\ker \rho_*)^{\perp}$. Then ρ is called Anti-Invariant Riemannian Submersion.*

The distribution $\ker \rho_*$ is integrable, it means that the manifold $\rho^{-1}(q)$, $q \in N$ of $\ker \rho_*$ is a total submanifold of M. From Definition 2.3, $J(\ker \rho_*)^{\perp} \cap \ker \rho_* \neq \{0\}$. If we represent the complementary orthogonal distribution to $J \ker \rho_*$ in $(\ker \rho_*)^{\perp}$ by μ. We obtain;

$$(\ker \rho_*)^{\perp} = J \ker \rho_* \oplus \mu.$$

Obviously μ is an invariant distribution of $(\ker \rho_*)^{\perp}$ with respect to the complex structure J. That is for any $F \in \Gamma(ker\rho_*)^{\perp}$,

$$JF = GF + CF, \tag{2.1}$$

where $GF \in \Gamma(\ker \rho_*)$ and $CF \in \Gamma(\mu)$. If μ does not exist, then ρ is called a *Lagrangian submersion.*

Definition 2.4. *[6] Let $\rho : (M, g^M, J) \rightarrow (N, g^N)$ be a Riemannian map from an Almost Hermitian manifold to a Riemannian manifold. Then ρ is called a Semi-Invariant Riemannian map if there are distributions $\mathcal{D}_1, \mathcal{D}_2 \subseteq \ker \rho_*$ such that*

$$\ker \rho_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \tag{2.2}$$

and

$$J\mathcal{D}_1 = \mathcal{D}_1, \quad J\mathcal{D}_2 \subseteq (\ker \rho_*)^{\perp}, \tag{2.3}$$

$\mathcal{D}_2$ is orthogonal complementary distribution to $\mathcal{D}_1$ in $\ker \rho_*$.

If we let μ be the orthogonal complementary distribution to $J \ker \rho_*$ in $(\ker \rho_*)^{\perp}$. Then

$$(\ker \rho_*)^{\perp} = J\mathcal{D}_2 \oplus \mu.$$

It is obvious that μ is an invariant distribution of $(\ker \rho_*)^{\perp}$, with respect to the complex structure J.

Definition 2.5. *[10] Let $\rho : (M, g^M, J) \to (N, g^N)$ be a Riemannian Submersion from an Almost Hermitian manifold onto a Riemannian manifold. If the angle $\theta(X)$ between the space $\ker \rho_{*p}$ and JX is a constant for any non-zero vector $X \in \ker \rho_{*p}$, $p \in M$, that is if θ is independent of the choice of the point $p \in M$ and choice of the tangent vector X in $\ker \rho_{*p}$. Then ρ is a slant-submersion.*

θ is called the *slant angle* of the slant submersion. Following the definition, the fibers of the slant submersion are slant submanifold of M. If $0 < \theta < \frac{\pi}{2}$ then the slant submersion is called a proper slant submersion.

If $\rho : (M, g^M, J) \to (N, g^N)$ is a slant submersion from an Almost Hermitian manifold onto a Riemannian manifold, then for any $X \in \Gamma(\ker \rho_*)$, we have;

$$JX = \phi X + \omega X,$$

ϕX is a vertical part and ωX is the horizontal parts of JX.

Similarly, given any $F \in \Gamma((\ker \rho_*)^\perp)$,

$$JF = GF + CF,$$

GF is the vertical component and CF is the horizontal component of JF. (see [10] for more detail).

Definition 2.6. *[11] A Riemannian Submersion $\rho : (M, g^M, J) \longrightarrow (N, g^N)$ from an Almost Hermitian manifold to a Riemannian manifold is called a Semi-Slant submersion if there are some distributions $\mathcal{D}_1, \mathcal{D}_2 \subset \ker \rho_*$ with the property that*

$$\ker \rho_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between the spaces $\mathcal{D}_2$ and JX is constant for any non-zero vector $X \in \mathcal{D}_2$. $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker \rho_*$.

Definition 2.7. *[15] Let (M, g^M, J) be an even dimensional Almost Hermitian manifold whose Hermitian metric is g^M and almost complex structure J^M, and $(N, g^N)N$ be a Riemannian manifold with Riemannian metric g^N. A Riemannian Submersion $\rho : (M, g^M, J) \to (N, g^N)$ is a Hemi-Slant submersion if the vertical distribution $\ker \rho_*$ of ρ has two orthogonal complementary distributions D^θ and $D^\perp$ such that D^θ is a slant distribution and $D^\perp$ is anti-invariant distribution, i.e*

$$\ker \rho_*) = D^\theta \oplus D^\perp,$$

The angle θ in the slant distribution is called the Hemi-Slant angle of the submersion.

3 Quasi-Hemi-Slant Submersion

Here we recall the modified submersion introduced by S.Longwap at el, which is the generalization of Hemi-Slant, Semi-Slant and Semi-Invarian Riemannian Submersion from Almost Hermitian manifold to a Riemannian manifold [16]. This type of Riemannian Submersion has been extended to other manifolds. See [23], [24],[25], [26], [27] and [28] for more details.

Theorem 3.1. *[16] Suppose that (M, g^M, J) is a $2m$-dimensional Almost Hermitian manifold, J is the complex structure and g^M is the metric in M. Let (N, g^N) be a Riemannian manifold with its Riemannian metric g^N. There exists a Riemannian Submersion $\rho : (M, g^M, J) \longrightarrow (N, g^N)$, where the vertical distribution $\ker \rho_*$ is a direct sum of three orthogonal distributions $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, such that $\mathcal{D}$ is invariant, $\mathcal{D}^\theta$ is slant and $\mathcal{D}^\perp$ is anti-invariant. That is;*

$$\ker \rho_* = \mathcal{D} \oplus \mathcal{D}^\theta \oplus \mathcal{D}^\perp, \tag{3.1}$$

with $J\mathcal{D} = \mathcal{D}$, the angle θ between $J\mathcal{D}^\theta$ and $\mathcal{D}^\theta$ is a constant and $J\mathcal{D}^\perp \subseteq (\ker \rho_*)^\perp$.

If the dimension of $\mathcal{D}$ is m_1, dimension of $\mathcal{D}^\theta$ is m_2 and the dimension of $\mathcal{D}^\perp$ is m_3, then we observe the following:

(a) If $m_1 = 0$, then $\ker \rho_* = \mathcal{D}^\theta \oplus \mathcal{D}^\perp$ and M is a Hemi-Slant submersion

(b) If $m_2 = 0$, then $\ker \rho_* = \mathcal{D} \oplus \mathcal{D}^\perp$, M is a Semi-Invariant submersion

(c) If $m_3 = 0$, then $\ker \rho_* = \mathcal{D} \oplus \mathcal{D}^\theta$, M is a Semi-Slant submersion

The Riemannian submersion in Theorem 3.1 is called *Quasi-Hemi-Slant* ($QHSS$) *submersion* and the angle θ is called the *Quasi-Hemi-Slant angle* ($QHSSA$) of the submersion. The Hemi-Slant, Semi-Invariant and Semi-Slant Riemannian submersions are special cases of the Quasi-Hemi-Slant submersion($QHSS$). The Quasi-Hemi-Slant submersion $\rho : (M, g^M, J) \longrightarrow (N, g^N)$ is *proper* Quasi-Hemi-Slant submersion if
$dim(\mathcal{D}) \neq 0$, $dim(\mathcal{D}^\perp) \neq 0$, $dim(\mathcal{D}^\theta) \neq 0$ and $0 < \theta < \frac{1}{2}$.

We observe that, Hemi-Slant submersions, Semi-Invariant submersions, and Semi-Slant submersions are all special cases of Quasi-Hemi-Slant submersion.

Let $\rho : (M, g^M, J) \to (N, g^N)$ be a Quasi-Hemi-Slant submersion from an Almost Hermitian manifold onto a Riemannian manifold. Then we know that the tangent bundle is a direct sum of its vertical distribution $\ker \rho_*$ and its horizontal distribution $(\ker \rho_*)^\perp$ that is,

$$TM = \ker \rho_* \oplus (\ker \rho_*)^\perp. \tag{3.2}$$

If we define the projections $\mathcal{P}$ and $\mathcal{Q}$ on the tangent bundle TM of M by $\mathcal{P} : TM \longrightarrow \ker \rho_*$ and $\mathcal{Q} : TM \longrightarrow (\ker \rho_*)^\perp$, respectively, then for all $X \in \Gamma(TM)$,

$$X = \mathcal{P}X + \mathcal{Q}X, \tag{3.3}$$

in this equation $\mathcal{P}X \in \Gamma(\ker \rho_*)$ and $\mathcal{Q}X \in \Gamma((\ker \rho_*)^\perp)$.

For any vector field $X \in \Gamma(\ker \rho_*)$, it can be written in the form

$$X = \mathcal{P}X + \mathcal{Q}X + \mathcal{R}X, \tag{3.4}$$

in this equation $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ are projections of $\ker \rho_*$ onto $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ respectively. Since we know that

$$JX = \phi X + \omega X, \tag{3.5}$$

for $\phi X \in \Gamma(\ker \rho_*)$ and $\omega X \in \Gamma((\ker \rho_*)^\perp)$. From (3.4) and (3.5), we obtain the equation

$$JX = \phi\mathcal{P}X + \omega\mathcal{P}X + \phi\mathcal{Q}X + \omega\mathcal{Q}X + \phi\mathcal{R}X + \omega\mathcal{R}X.$$

Since $J\mathcal{D} = \mathcal{D}$ and $J\mathcal{D}^\perp \subseteq (\ker \rho_*)^\perp$, we have $\omega PX = 0$ and $\phi RX = 0$, and so

$$JX = \phi\mathcal{P}X + \phi\mathcal{Q}X + \omega\mathcal{Q}X + \omega\mathcal{R}X. \tag{3.6}$$

This shows that

$$J \ker \rho_* = \mathcal{D} \oplus \phi\mathcal{D}^\theta \oplus \omega\mathcal{D}^\theta \oplus J\mathcal{D}^\perp. \tag{3.7}$$

Since $\omega\mathcal{D}^\theta \subseteq (\ker \rho_*)^\perp$ and $J\mathcal{D}^\perp \subseteq (\ker \rho_*)^\perp$, we obtain

$$(\ker \rho_*)^\perp = \omega\mathcal{D}^\theta \oplus J\mathcal{D}^\perp \oplus \mu, \tag{3.8}$$

μ is the orthogonal complement of $\omega\mathcal{D}^\theta \oplus J\mathcal{D}^\perp$ in $(\ker \rho_*)^\perp$. The distribution μ is invariant with respect to complex structure J. Similarly, given any vector $F \in (\ker \rho_*)^\perp$, we can write

$$JF = \mathcal{G}F + \mathcal{C}F. \tag{3.9}$$

where $\mathcal{G}F \in \Gamma(\omega\mathcal{D}^\theta \oplus J\mathcal{D}^\perp)$ and that $\mathcal{C}F \in \Gamma(\mu)$.

Therefore,from the equations (3.4), (3.8) and (3.9), we get the following.

Lemma 3.2. *Let $\rho : (M, g^M, J) \to (N, g^N)$ be a Quasi-Hemi-Slant submersion from an Almost Hermitian manifold onto a Riemannian manifold. Then the following equtions hold;*

$$\phi\mathcal{D}^\theta = \mathcal{D}^\theta, \;\; \phi\mathcal{D}^\perp = \{0\}, \;\; \mathcal{G}\omega\mathcal{D}^\theta = \mathcal{D}^\theta, \;\; \mathcal{G}\omega\mathcal{D}^\perp = \mathcal{D}^\perp.$$

On the other hand, if we compare the normal and the tangential components of equations (3.4), (3.9) together with the fact that $J^2 = -\mathbb{I}$, the following results are obtain

Theorem 3.3. *Let ϕ and ω, $\mathcal{G}$ and $\mathcal{C}$ be endomorphisms in the tangent bundle of the manifold (M, g^M, J). Then the following equations hold;*

(i) $\phi^2 + \mathcal{G}\omega = -\mathbb{I}$, *(ii)* $\omega\phi + \mathcal{C}\omega = 0$,

(iii) $\omega\mathcal{G} + \mathcal{C}^2 = -\mathbb{I}$, *(iv)* $\phi\mathcal{G} + \mathcal{G}\mathcal{C} = 0$,

in these equations, $\mathbb{I}$ represents the identity operator on the total space of ρ.

Proof. This result is from Lemma 3.2. □

Lemma 3.4. *Let $\rho : (M, g^M, J) \to (N, g^N)$ be a proper Quasi-Hemi-Slant submersion from an Almost Hermitian manifold onto a Riemannian manifold. Then we have the following;*

$$\cos^2\theta X = -\phi^2 X, \tag{3.10}$$

for all vector fields X, $Y \in \Gamma(\mathcal{D}^\theta)$, θ is the Quasi-Hemi-Slant angle and ϕ is the endomorphim in theorem 3.3

Proof. We let θ be Quasi-Hemi-Slant angle between JX and ϕX. Then
$\cos\theta = \dfrac{\|\phi X\|}{\|JX\|}$, $\cos\theta = \dfrac{g(JX, \phi X)}{\|JX\|\|\phi X\|}$
so these give $\cos^2\theta = \dfrac{g(JX, \phi X)}{\|JX\|^2}$.
From the properties of the complex structure J it follows that;

$\cos^2\theta = -\dfrac{g(X, \phi^2 X)}{g(JX, JX)}$, $\cos^2\theta = -\phi^2\dfrac{g(X, X)}{g(X, X)}$,
which shows that

$$\cos^2\theta = -\phi^2 \tag{3.11}$$

□

Lemma 3.5. *Let $\rho : (M, g^M, J_m) \to (M, g^N)$ be a Quasi-Hemi-Slant submersion from an Almost Hermitian manifold onto a Riemannian manifold. If θ is the Quasi-Hemi-Slant angle between JX and ϕX, $X \in \mathcal{D}^\theta \subset \ker\rho_*$. Then the following equations hold;*

1. *$g(\phi X, \phi X) = \cos^2\theta g(X, X)$ for all $X \in D^\theta$*
2. *$g(\omega X, \omega X) = \sin^2\theta g(X, X)$ for all $X \in D^\theta$*
3. *$g(\omega X, \omega X) = g(X, X)$ for every $X \in D^\perp$*

Proof. We assume θ is the angle between JX and ϕX. Then,

1. $\cos\theta = \frac{\|\phi X\|}{\|JX\|}, \quad \cos^2\theta = \frac{\|\phi X\|^2}{\|JX\|^2} \quad \cos^2\theta = \frac{g(\phi X, \phi X)}{g(JX, JX)}$

   so that $g(X, X)\cos^2\theta = g(\phi X, \phi X)$

   Similarly

2. $\sin\theta) = \frac{\|\omega X\|}{\|JX\|}, \quad \sin^2(\theta) = \frac{\|\omega X\|^2}{\|JX\|^2} \quad \sin^2\theta = \frac{g(\omega X, \omega X)}{g(JX, JX)}$

   and so $g(X, X)\sin^2\theta = g(\omega X, \omega X)$

3. If $JX = \omega X$ for every $X \in D^{\perp}$, then

$$g(X, X) = g(JX, JX) = g(\omega X, \omega X). \tag{3.12}$$

□

Theorem 3.6. *If $\rho : (M, g^M, J_m) \to (M, g^N)$ is a proper Quasi-Hemi-Slant submersion from an Almost Hermitian manifold onto a Riemannian manifold and suppose $\{e_1, e_2, ..., e_{m-n}\}$ is the local orthonormal basis of $ker\rho_*$. Then*

1. *$\{\sec\theta\phi e_1, \sec\theta\phi e_2, ..., \sec\theta\phi e_{(m-n)}\}$ is a local orthonormal basis of $\phi\mathcal{D}^\theta$.*
2. *$\{\csc\theta\phi e_1, \csc\theta\phi e_2, ..., \csc\theta\phi e_{(m-n)}\}$ is a local orthonormal basis of $\omega\mathcal{D}^\theta$.*

Proof. 1. In this proof, we only need to show that $g^M(\sec\theta\phi e_i, \sec\theta\phi e_j) = \delta_{ij}$, for all $i, j \in \{1, 2, ..., m-n\}$. We have from Lemma 3.5 that

$$\begin{aligned} g^M(\sec\theta\phi e_i, \sec\theta\phi e_j) &= \sec^2\theta\phi^2 g^M(e_i, e_j) \\ &= \sec^2\theta\cos^2\theta g^M(e_i, e_j) = g^M(e_i, e_j) = \delta_{ij} \end{aligned} \tag{3.13}$$

2. in a similarly way we show that $g^M(\csc\theta\omega e_i, \csc\theta\omega e_j) = \delta_{ij}$.

$$\begin{aligned} g^M(\csc\theta\omega e_i, \csc\theta\omega e_j) &= \csc^2\theta\omega^2 g_1(e_i, e_j) \\ &= \csc^2\theta\sin^2\theta g^M(e_i, e_j) = g^M(e_i, e_j) = \delta_{ij} \end{aligned} \tag{3.14}$$

as required. □

Corollary 3.7. *Let $\rho : (M, g^M, J_m) \to (M, g^N)$ be a Quasi-Hemi-Slant Riemannian submersion from an almost Hermitian manifold onto a Riemannian manifold. Let $\{e_1, e_2, ..., e_{m-n}\}$ be a local orthonormal basis of $\ker\rho_*$. Then*

1. *$\{e_1, \sec\theta\phi e_1, e_2, \sec\theta\phi e_2, ..., e_{m-n}, \sec\theta\phi e_{m-n}\}$ is a local orthonormal basis of $\phi(\ker\rho_*)$.*
2. *$\{\csc\theta\omega e_1, \omega e_1, \csc\theta\omega e_2, \omega e_2, ..., \csc\theta\omega e_{(m-n)}, \omega e_{(m-n)}\}$ is a local orthonormal basis of $\omega(\ker\rho_*)$.*

Proof. 1. We are to show that $g^M(\sec\theta\phi e_i, \sec\theta\phi e_j) = \delta_{ij}$. this result comes from theorem 3.6 and already $g^M(e_i, e_j) = \delta_{ij}$

2. Similarly, we show that $g^M(\csc\theta\phi e_i, \csc\theta\phi e_j) = \delta_{ij}$ and $g(\omega e_i, \omega e_j) = \delta_{ij}$. This is already in theorem 3.6 and from Lemma 3.5 we have

$$\begin{aligned} g^M(\omega e_i, \omega e_j) &= g^M(e_i, e_j) \\ &= \delta_{ij} \end{aligned}$$

and this ends the proof. □

4 Fundamental Tensor Fields

Recall that the tangent space TM of a Manifold M is split into horizontal and vertical subspaces. That is

$$TM = \mathcal{V}TM \oplus \mathcal{H}TM,$$

$\mathcal{H}TM$ are the horizontal vectors and $\mathcal{V}TM$ are the vertical vectors of the tangent space TM. Riemannian submersion $\pi : (M, g^M) \to (N, g^N)$ determines two $(1,2)-$tensor fields $\mathcal{T}$ and $\mathcal{A}$ called the fundamental tensor fields of π. The tensor fileds are defined by the horizontal projection $\mathcal{H} : TM \to \mathcal{H}TM$ and vertical projection $\mathcal{V} : TM \to \mathcal{V}TM$ according to the formula:

$$\begin{aligned} \mathcal{A}_E F &= \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \\ \mathcal{T}_E F &= \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \end{aligned} \quad (4.1)$$

for any vector fields E, F on M, where ∇ is the Levi-Civita connection of g^M.See [3]. $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle TM. If the vertical and the horizontal vectors are reversed, then we can now summarize the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$. Let α, β be vertical and ξ, η be horizontal vector fields on M, then we have

$$\begin{aligned} \nabla_\alpha \beta &= \mathcal{T}_\alpha \beta + \mathcal{V}\nabla_\alpha \beta, \\ \nabla_\alpha \xi &= \mathcal{T}_\alpha \xi + \mathcal{H}\nabla_\alpha \xi, \\ \nabla_\xi \alpha &= \mathcal{A}_\xi \alpha + \mathcal{V}\nabla_\xi \alpha, \\ \nabla_\xi \eta &= \mathcal{A}_\xi \eta + \mathcal{H}\nabla_\xi \eta + \end{aligned} \quad (4.2)$$

We observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, and $\mathcal{A}$ acts on the horizontal distribution as measures of the obstruction to integrability of the distribution. For details on fundamental tensor fields, we refer to [1].

Let (M, g^M, J) be hermitian manifold with the complex structure J, and (N, g^N) be a Riemannian manifold. The complex structure effects the fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$ of the submersion.

Proposition 4.1. *[16] Let $\rho : (M, g^M, J) \to (N, g^N)$ be a quasi-hemi-slant submersion from a Kaehler manifold onto a Riemannian manifold. Then we have the following equations:*

$$\begin{aligned} \mathcal{V}\nabla_\alpha \phi + \mathcal{T}_\alpha \omega\beta &= \phi\mathcal{V}\nabla_\alpha \beta + \mathcal{G}\mathcal{T}_\alpha \beta, \quad (4.3) \\ \mathcal{T}_\alpha \phi\beta + \mathcal{H}\nabla_\alpha \omega\beta &= \omega\mathcal{V}\nabla_\alpha \beta + \mathcal{C}\mathcal{T}_\alpha \beta, \\ \mathcal{V}\nabla_\xi \eta + \mathcal{A}_\xi \mathcal{C}\eta &= \phi\mathcal{A}_\xi \eta + \mathcal{G}\mathcal{H}\nabla_\xi \eta, \\ \mathcal{A}_\xi \mathcal{G}\eta + \mathcal{H}\nabla_\xi \mathcal{C}\eta &= \omega\mathcal{A}_\xi \eta + \mathcal{C}\mathcal{H}\nabla_\xi \eta, \\ \mathcal{V}\nabla_\alpha \mathcal{G}\xi + \mathcal{T}_\alpha \mathcal{C}\xi &= \phi\mathcal{T}_\alpha \xi + \mathcal{G}\mathcal{H}\nabla_\alpha \xi, \\ \mathcal{T}_\alpha \mathcal{G}\xi + \mathcal{H}\nabla_\alpha \mathcal{C}\xi &= \omega\mathcal{T}_\alpha \xi + \mathcal{C}\mathcal{H}\nabla_\alpha \xi, \quad (4.4) \end{aligned}$$

where α, $\beta \in \Gamma(\ker\rho_*)$ and ξ, $\eta \in (\ker\pi_*)^\perp$, $\mathcal{G}$ and $\mathcal{C}$ are projections of vectors in $\Gamma(\ker\rho_*)^\perp$ to vertical and horizontal vectors respectively while $\mathcal{V}$ and $\mathcal{H}$ are projections of vectors in $\Gamma(\ker\rho_*)$ to vertical and horizontal vectors respectively.

Corollary 4.1. *If ρ is a Quasi-Hemi-Slant submersion from a manifold (M, g^M, J) onto a Riemannian manifold (N, g^N), then*

$$\begin{aligned}(\nabla_X\phi)Y &:= \mathcal{V}\nabla_X\phi Y - \phi\mathcal{V}\nabla_X Y,\\ (\nabla_X\omega)Y &:= \mathcal{H}\nabla_X\omega Y - \omega\mathcal{V}\nabla_X Y,\\ (\nabla_Z\mathcal{C})W &:= \mathcal{H}\nabla_Z\mathcal{C}W - \mathcal{C}\mathcal{H}\nabla_Z W,\\ (\nabla_Z\mathcal{G})W &:= \mathcal{V}\nabla_Z\mathcal{G}W - \mathcal{G}\mathcal{H}\nabla_Z W, \end{aligned} \tag{4.5}$$

for all $X, Y \in \Gamma(\ker\pi_*)$ and $Z, W \in \Gamma((\ker\pi_*)^\perp)$.

Thus, we have the following:

Corollary 4.2. *Let ρ be a quasi-hemi-slant submersion from a Kaehler manifold (M, g^M, J) onto a Riemannian manifold (N, g^N). Then we have*

(i) $(\nabla_X\phi)Y = \mathcal{G}\mathcal{T}_X Y - \mathcal{T}_X\omega Y$,

(ii) $(\nabla_X\omega)Y = \mathcal{C}\mathcal{T}_X Y - \mathcal{T}_X\phi Y$,

(iii) $(\nabla_Z\mathcal{C})W = \omega\mathcal{A}_Z W - \mathcal{A}_Z\mathcal{G}W$,

(iv) $(\nabla_Z\mathcal{G})W = \phi\mathcal{A}_Z W - \mathcal{A}_Z\mathcal{C}W$,

for all vectors $X, Y \in \Gamma(\ker\pi_*)$ and $Z, W \in \Gamma((\ker\pi_*)^\perp)$.

If the tensors ϕ and ω are parallel with respect to the linear connection ∇ on M, then

$$\mathcal{G}\mathcal{T}_X Y = \mathcal{T}_X\omega Y,$$

and

$$\mathcal{C}\mathcal{T}_X Y = \mathcal{T}_X\phi Y,$$

for all vectors X, $Y \in \Gamma(TM)$.

For the integrability of distributions and decomposition theorem on the Quasi-Hemi-Slant submersion, we refer to [16] for the details.

5 Conclusions

Let (M, g^M, J) be a $2m$-dimensional Almost Hermitian manifold with g^M a Riemannian metric on M and almost complex structure J, and (N, g^N) be a Riemannian manifold with Riemannian metric g^N. Then there is a Riemannian Submersion $\rho : (M, g^M, J) \longrightarrow (N, g^N)$ such that its vertical distribution $\ker\rho_*$ admits three orthogonal distributions $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ which are invariant, slant and anti-invariant respectively.Let ρ be a Quasi-Hemi-Slant submersion submersion from an almost Hermitian manifold (M, g^m, J) onto a Riemannian manifold (M, g^n). Let $\{e_1, e_2, ..., e_{m-n}\}$ be a local orthonormal basis of $ker\rho_*$. Then

1. $\{e_1, \sec\theta\phi e_1, e_2, \sec\theta\phi e_2, ..., e_{m-n}, \sec\theta\phi e_{m-n}\}$ is a local orthonormal basis of $\phi(ker\rho_*)$.
2. $\{csc\theta\omega e_1, \omega e_1, csc\theta\omega e_2, \omega e_2, ..., csc\theta\omega e_{(m-n)}, \omega e_{(m-n)}\}$ is a local orthonormal basis of $\omega(ker\rho_*)$.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] O'Neill B. The fundamental equations of a submersion. Mich. Math. J. 1966;13:458–469.

[2] Gray A. Pseudo-Riemannian almost product manifolds and submersions. J. Math. Mech. 1967;16:715-738.

[3] Falcitelli, Maria et al. Riemannian Submersions and related topics. World Scientific; 2004.

[4] Watson B. Almost Hermitian submersions. J. Differential Geom. 1976;11(1):147-165.

[5] Bayram Sahin. Riemannian submersion fron almost hermitian manifolds. TAIWANESE Journal of Mathematic. 2013;17(2):629-659.

[6] Sahin, Bayram. Semi-Invarian Riemannian maps fron Almost Hermitian manifolds. Ndagationes Mathematicae Elsevier. 2012;23:80-94.

[7] Sahin, Bayram. Anti-invariant riemannian submersions from almost Hermitian manifolds. Open Mathematics. 2010;8(3):437-447.

[8] Shahid, Mohammad Hasan, et al. Submersion of semi-invarian submanifolds of Trans-Sasakian manifold. Asian Academy of Management Journal of Accounting & Finance. 2013;9(1).

[9] Chen BY. Geometry of slant submaniflods. Katholieke Universiteit Leuven, Leuven.1990.

[10] Sahin B. Slant submersions from Almost Hermitian manifolds. Bull. Math. Soc. Sci. Math. Roumanie Tome. 2011;54(102):93-105.

[11] Park, Kwang-Soon, Prasad, Rajendra. Semi-Slant submersions. Bull. Korean Math. Soc. 2013;3:951-962.

[12] Ianus S, Mazzocco R, Vilcu GE. Riemannian submersions from quaternionic manifolds. Acta. Appl. Math. 2008;104:83-89.

[13] Ianus S, Ionescu AM, Mazzocco R, Vilcu GE. Riemannian submersions from almost contact metric manifolds.
arXiv: 1102.1570v1 [math. DG]

[14] Park KS. H-slant submersions, Bull. Korean Math. Soc. 2012;49(2):329-338.

[15] Tastan, Hakan Mete, et al. Hemi-Slant submersions. Mediterr J Math. 2016;10(4):s00009-015.

[16] Longwap S, Massamba F, Homti NE. On quasi-hemi-slant riemannian submersion. Journal of Advances in Mathematics and Computer Science. 2019;34(1):1-14.

[17] Bourguignon JP, Lawson HB. A mathematician's visit to Kaluza-Klein theory. Rend. Sem. Mat. Univ. Politec. Torino. 1989;(Special Issue):143-163.

[18] Ianus S, Visinescu M. Kaluza-Klein theory with scalar fields and generalized Hopf manifolds. Class. Quantum Gravity. 1987;4:1317-1325.

[19] Bourguignon JP, Lawson HB. Stability and isolation phenomena for Yang-mills fields. Commum. Math. Phys. 1981;79:189-230.

[20] B Watson, G. $G'-$Riemannian Submersions and nonlinear gauge field equations of general relativity, in: Global Analysis - Analysis on manifolds, T. Rassias, (ed.) dedicated M. Morse. Teubner-Texte Math. Teubner, Leipzig. 1983;57:324-349.

[21] Ianus S, Visinescu M. Space-time compactification and Riemannian submersions. in: The Mathematical Heritage of C. F. Gauss, G. Rassias, (ed.). World Scientific, River Edge. 1991;358-371.

[22] Mustafa MT. Applications of harmonic morphisms to gravity. J. Math. Phys. 2000;41(10):6918-6929.

[23] Prasad, Rajendra, at el. Quasi Bi-slant submanifolds of Kaehler manifolds. International Electronic Journal of Geometry. 2022;15(1):57-68.

[24] Hakami, Ali H at el. Aspects of submanifolds on (α, β)-type almost contact manifolds with quasi-hemi-slant factor. Symmetry. 2023;15(6):1270.

[25] Khan T, at el. Quasi hemi-slnat submanifolds of Quasi-Sasakian manifolds. Acta Univ. Apulensis. 2021;67:19-35.

[26] Rawat, Pramod Kumar, Kumar, Sushil. On quasi hemi-slant submersions. Communications in Advanced Mathematical Sciences. 2023;6(2):86-97.

[27] Akyol, Mehmet Akif, Prasad, Rajendra. Semi-Slant ξ-, Hemi-Slant ξ-Riemannian submersions and quasi hemi-slant submanifolds. Contact Geometry of Slant Submanifolds (Springer). 2022;301-332.

[28] Karmakar, Payel, Bhattacharyya, Arindam. Quasi hemi-slant submanifolds of trans-sasakian manifold. Palest J Math. 2023;12(1):745-756.

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