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The Convolution Sums $\sum_{m<\frac{n}{2}} m^2\sigma_e(m)\sigma_f(n-2m)$ and Its Applications

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Abstract

In this paper we obtain the convolution sum formulae of

$$\sum_{m<\frac{n}{2}} m^2\sigma_e(m)\sigma_f(n-2m)$$

for $n(\in \mathbb{N})$ and an odd positive integer e and f. Moreover we obtain some identities induced from $\sum_{m<\frac{n}{2}} m^2\sigma_e(m)\sigma_f(n-2m)$.

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1 Introduction

For $n, m \in \mathbb{N}$, $s, r \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define some necessary divisor functions and infinite product sums which also appear in many areas of number theory:

$$\sigma_s(n) = \sum_{d|n} d^s, \qquad \sigma_{s,r}(n;m) = \sum_{\substack{d|n \\ d\equiv r \bmod m}} d^s,$$
$$\Delta(q) := \sum_{n=1}^{\infty} \tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24},$$
$$B(q) := \sum_{n=1}^{\infty} b(n)q^n = (\Delta(q)\Delta(q^2))^{\frac{1}{3}} = q\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^8, \tag{1.1}$$
$$C(q) := \sum_{n=1}^{\infty} c(n)q^n = (\Delta(q)^4\Delta(q^2))^{\frac{1}{6}} = q\prod_{n=1}^{\infty}(1-q^n)^{16}(1-q^{2n})^4,$$
$$D(q) := \sum_{n=1}^{\infty} d(n)q^n = (\Delta(q)^2\Delta(q^2)\Delta(q^4)^2)^{\frac{1}{6}} = q^2\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^4(1-q^{4n})^8.$$

In general, it is satisfied that
$$b(n) = -8b(\frac{n}{2}) \tag{1.2}$$
for even n (see [1], Remark 4.3). We note that

$$\sigma_s(n) = \sigma_{s,1}(n;2) + \sigma_{s,0}(n;2) \tag{1.3}$$

and

$$\sigma_{s,1}(2n;2) = \sigma_{s,1}(n;2), \quad \sigma_{s,0}(2n;2) = 2^s\sigma_s(n). \tag{1.4}$$

The Eisenstein series $L(q)$, $M(q)$, and $N(q)$ for $q \in \mathbb{C}$ with $|q| < 1$ are

$$L(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n, \tag{1.5}$$

$$M(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{1.6}$$

$$N(q) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n \tag{1.7}$$

by ([2], p. 318). Lahiri ([3], p. 149) has derived the following identities from the work of Ramanujan [4] :

$$L^2(q) = 1 - 288\sum_{n=1}^{\infty}n\sigma_1(n)q^n + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{1.8}$$

$$L^3(q) = 1 - 1728\sum_{n=1}^{\infty}n^2\sigma_1(n)q^n + 2160\sum_{n=1}^{\infty}n\sigma_3(n)q^n - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{1.9}$$

$$L(q)M(q) = 1 + 720\sum_{n=1}^{\infty}n\sigma_3(n)q^n - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{1.10}$$

$$L^2(q)M(q) = 1 + 1728\sum_{n=1}^{\infty} n^2\sigma_3(n)q^n - 2016\sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n, \tag{1.11}$$

$$L(q)N(q) = 1 - 1008\sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n, \tag{1.12}$$

$$L^2(q)N(q) = 1 - 1728\sum_{n=1}^{\infty} n^2\sigma_5(n)q^n + 1440\sum_{n=1}^{\infty} n\sigma_7(n)q^n - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n. \tag{1.13}$$

For $e, f, m, n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$ we define

$$\begin{aligned} I^k_{m,e,f}(n) &:= \sum_{m=1}^{n-1} m^k\sigma_e(m)\sigma_f(n-m), \\ I_{m,e,f}(n) := I^1_{m,e,f}(n) &= \sum_{m=1}^{n-1} m\sigma_e(m)\sigma_f(n-m), \\ I_{e,f}(n) := I^0_{m,e,f}(n) &= \sum_{m=1}^{n-1}\sigma_e(m)\sigma_f(n-m). \end{aligned} \tag{1.14}$$

Ramanujan [4] and Lahiri [3], [5] have shown that $I_{e,f}$ can be expressed as :

$$\begin{aligned}
I_{1,1}(n) &= \frac{5}{12}\sigma_3(n) + \frac{(1-6n)}{12}\sigma_1(n), \\
I_{1,3}(n) &= \frac{7}{80}\sigma_5(n) + \frac{(1-3n)}{24}\sigma_3(n) - \frac{1}{240}\sigma_1(n), \\
I_{1,5}(n) &= \frac{5}{126}\sigma_7(n) + \frac{(1-2n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(n), \\
I_{3,3}(n) &= \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n), \\
I_{1,7}(n) &= \frac{11}{480}\sigma_9(n) + \frac{(2-3n)}{48}\sigma_7(n) - \frac{1}{480}\sigma_1(n), \\
I_{3,5}(n) &= \frac{11}{5040}\sigma_9(n) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(n), \\
I_{1,9}(n) &= \frac{455}{30404}\sigma_{11}(n) + \frac{(5-6n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1(n) - \frac{36}{3455}\tau(n), \\
I_{3,7}(n) &= \frac{91}{110560}\sigma_{11}(n) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(n) + \frac{15}{2764}\tau(n), \\
I_{5,5}(n) &= \frac{65}{174132}\sigma_{11}(n) + \frac{1}{252}\sigma_5(n) - \frac{3}{691}\tau(n), \\
I_{1,11}(n) &= \frac{691}{65520}\sigma_{13}(n) + \frac{(1-n)}{24}\sigma_{11}(n) - \frac{691}{65520}\tau(n), \\
I_{3,9}(n) &= \frac{1}{2640}\sigma_{13}(n) - \frac{1}{240}\sigma_9(n) + \frac{1}{264}\sigma_3(n), \\
I_{5,7}(n) &= \frac{1}{10080}\sigma_{13}(n) + \frac{1}{504}\sigma_7(n) - \frac{1}{480}\sigma_5(n).
\end{aligned}$$

And we have already obtained :

Proposition 1.1. *Let $n \in \mathbb{N}$. Then we have*

(a) *(See ([6], p. 155))*

$$I_{m,1,1}(n) = \frac{1}{24} n \left\{ 5\sigma_3(n) - (6n-1)\,\sigma_1(n) \right\},$$

(b) *(See ([6], p. 157))*

$$I_{m,1,3}(n) = \frac{7}{240} n\sigma_5(n) - \frac{1}{40} n^2\sigma_3(n) - \frac{1}{240} n\sigma_1(n),$$
$$I_{m,3,1}(n) = \frac{7}{120} n\sigma_5(n) + \left(\frac{1}{24} n - \frac{1}{10} n^2 \right) \sigma_3(n),$$

(c) *(See ([7], Theorem 3.3, Theorem 3.1), ([6], p. 155))*

$$I_{m,1,5}(n) = \frac{1}{504} n \left\{ 5\sigma_7(n) - 6n\sigma_5(n) + \sigma_1(n) \right\},$$
$$I_{m,3,3}(n) = \frac{1}{240} n \left\{ \sigma_7(n) - \sigma_3(n) \right\},$$
$$I_{m,5,1}(n) = \frac{1}{168} n \left\{ 5\sigma_7(n) - (12n-7)\,\sigma_5(n) \right\},$$

(d) *(See ([7], Theorem 3.8 (a), (b), Theorem 3.7, Theorem 3.5))*

$$I_{m,1,7}(n) = \frac{1}{7200} \left\{ 33n\sigma_9(n) - 50n^2\sigma_7(n) - 15n\sigma_1(n) + 32\tau(n) \right\},$$
$$I_{m,3,5}(n) = \frac{1}{12600} \left\{ 11n\sigma_9(n) + 25n\sigma_3(n) - 36\tau(n) \right\},$$
$$I_{m,5,3}(n) = \frac{1}{8400} \left\{ 11n\sigma_9(n) - 35n\sigma_5(n) + 24\tau(n) \right\},$$
$$I_{m,7,1}(n) = \frac{1}{1800} \left\{ 33n\sigma_9(n) - 25n\,(4n-3)\,\sigma_7(n) - 8\tau(n) \right\}.$$

Next we can see the following proposition :

Proposition 1.2. *Let $n \in \mathbb{N}$. Then we have*

(a) *(See ([6], p. 155))*

$$I^2_{m,1,1}(n) = \frac{1}{8} n^2\sigma_3(n) + \left(\frac{1}{24} n^2 - \frac{1}{6} n^3 \right) \sigma_1(n),$$

(b) *(See ([6], p. 155∼156))*

$$I^2_{m,1,3}(n) = \frac{1}{80} n^2\sigma_5(n) - \frac{1}{120} n^3\sigma_3(n) - \frac{1}{240} n^2\sigma_1(n),$$
$$I^2_{m,3,1}(n) = \frac{1}{24} n^2\sigma_5(n) + \left(\frac{1}{24} n^2 - \frac{1}{12} n^3 \right) \sigma_3(n),$$

(c) *(See ([8], Theorem 1.1 (a), (b)))*

$$I^2_{m,1,5}(n) = \frac{1}{3024} \left\{ 10n^2\sigma_7(n) - 9n^3\sigma_5(n) + 6n^2\sigma_1(n) - 7\tau(n) \right\},$$
$$I^2_{m,3,3}(n) = \frac{1}{2160} \left\{ 5n^2\sigma_7(n) - 9n^2\sigma_3(n) + 4\tau(n) \right\}.$$

Similarly, for $e, f, m, n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$ we set

$$
\begin{aligned}
T^k_{m,e,f}(n) &:= \sum_{m<\frac{n}{2}} m^k \sigma_e(m)\sigma_f(n-2m), \\
T_{m,e,f}(n) := T^1_{m,e,f}(n) &= \sum_{m<\frac{n}{2}} m\sigma_e(m)\sigma_f(n-2m), \\
T_{e,f}(n) := T^0_{m,e,f}(n) &= \sum_{m<\frac{n}{2}} \sigma_e(m)\sigma_f(n-2m).
\end{aligned}
\tag{1.15}
$$

In 1997, Melfi considered especially the convolution sums $T_{1,1}(n)$, $T_{1,3}(n)$, $T_{3,1}(n)$, and for odd n, Melfi proved that

$$
\begin{aligned}
T_{1,1}(n) &= \frac{1}{12}\sigma_3(n) + \frac{(1-3n)}{24}\sigma_1(n), \quad \text{if} \quad n \equiv 1 \pmod 2, \\
T_{1,3}(n) &= \frac{1}{48}\sigma_5(n) + \frac{(2-3n)}{48}\sigma_3(n), \quad \text{if} \quad n \equiv 1 \pmod 2, \\
T_{3,1}(n) &= \frac{1}{240}\sigma_5(n) - \frac{1}{240}\sigma_1(n), \quad \text{if} \quad n \equiv 1 \pmod 2.
\end{aligned}
$$

In 2002 Huard, Ou, Spearman and Williams [9] evaluated $T_{1,1}(n)$, $T_{1,3}(n)$, and $T_{3,1}(n)$ for $n \in \mathbb{N}$, that is,

$$
\begin{aligned}
T_{1,1}(n) &= \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(\frac{n}{2}) + \frac{(1-3n)}{24}\sigma_1(n) + \frac{(1-6n)}{24}\sigma_1(\frac{n}{2}), \\
T_{1,3}(n) &= \frac{1}{48}\sigma_5(n) + \frac{1}{15}\sigma_5(\frac{n}{2}) + \frac{(2-3n)}{48}\sigma_3(n) - \frac{1}{240}\sigma_1(\frac{n}{2}), \\
T_{3,1}(n) &= \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5(\frac{n}{2}) + \frac{(1-3n)}{24}\sigma_3(\frac{n}{2}) - \frac{1}{240}\sigma_1(n).
\end{aligned}
$$

For the other $T_{e,f}(n)$ we refer to ([10], p. 45–54) :

$$
\begin{aligned}
T_{5,1}(n) &= \frac{1}{2142}\sigma_7(n) + \frac{2}{51}\sigma_7(\frac{n}{2}) + \frac{(1-2n)}{24}\sigma_5(\frac{n}{2}) + \frac{1}{504}\sigma_1(n) - \frac{1}{408}b(n),\\
T_{3,3}(n) &= \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7(\frac{n}{2}) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{2}) + \frac{1}{272}b(n),\\
T_{1,5}(n) &= \frac{1}{102}\sigma_7(n) + \frac{32}{1071}\sigma_7(\frac{n}{2}) + \frac{(1-n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(\frac{n}{2}) - \frac{1}{102}b(n),\\
T_{7,1}(n) &= \frac{1}{14880}\sigma_9(n) + \frac{17}{744}\sigma_9(\frac{n}{2}) + \frac{(2-3n)}{48}\sigma_7(\frac{n}{2}) - \frac{1}{480}\sigma_1(n)\\
&\quad + \frac{1}{496}c(n) + \frac{2}{31}d(n),\\
T_{5,3}(n) &= \frac{1}{31248}\sigma_9(n) + \frac{1}{465}\sigma_9(\frac{n}{2}) - \frac{1}{240}\sigma_5(\frac{n}{2}) + \frac{1}{504}\sigma_3(n)\\
&\quad - \frac{1}{496}c(n) - \frac{2}{31}d(n),\\
T_{3,5}(n) &= \frac{1}{7440}\sigma_9(n) + \frac{4}{1953}\sigma_9(\frac{n}{2}) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(\frac{n}{2})\\
&\quad + \frac{1}{248}c(n) + \frac{4}{31}d(n),\\
T_{1,7}(n) &= \frac{17}{2976}\sigma_9(n) + \frac{8}{465}\sigma_9(\frac{n}{2}) + \frac{(4-3n)}{96}\sigma_7(n) - \frac{1}{480}\sigma_1(\frac{n}{2})\\
&\quad - \frac{1}{62}c(n) - \frac{16}{31}d(n),\\
T_{9,1}(n) &= \frac{1}{91212}\sigma_{11}(n) + \frac{31}{2073}\sigma_{11}(\frac{n}{2}) + \frac{(5-6n)}{120}\sigma_9(\frac{n}{2}) + \frac{1}{264}\sigma_1(n)\\
&\quad - \frac{21}{5528}\tau(n) - \frac{282}{3455}\tau(\frac{n}{2}),\\
T_{7,3}(n) &= \frac{1}{331680}\sigma_{11}(n) + \frac{17}{20730}\sigma_{11}(\frac{n}{2}) - \frac{1}{240}\sigma_7(\frac{n}{2}) - \frac{1}{480}\sigma_3(n)\\
&\quad + \frac{23}{11056}\tau(n) + \frac{91}{1382}\tau(\frac{n}{2}),\\
T_{5,5}(n) &= \frac{1}{174132}\sigma_{11}(n) + \frac{16}{43533}\sigma_{11}(\frac{n}{2}) + \frac{1}{504}\sigma_5(n) + \frac{1}{504}\sigma_5(\frac{n}{2})\\
&\quad - \frac{11}{5528}\tau(n) - \frac{88}{691}\tau(\frac{n}{2}),\\
T_{3,7}(n) &= \frac{17}{331680}\sigma_{11}(n) + \frac{8}{10365}\sigma_{11}(\frac{n}{2}) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(\frac{n}{2})\\
&\quad + \frac{91}{22112}\tau(n) + \frac{368}{691}\tau(\frac{n}{2}),\\
T_{1,9}(n) &= \frac{31}{8292}\sigma_{11}(n) + \frac{256}{22803}\sigma_{11}(\frac{n}{2}) + \frac{(5-3n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1(\frac{n}{2})\\
&\quad - \frac{141}{6910}\tau(n) - \frac{2688}{691}\tau(\frac{n}{2}).
\end{aligned}
$$

Also we can find

$$
T_{m,1,1}(n) = \frac{1}{48}n\sigma_3(n) - \frac{1}{48}n^2\sigma_1(n) + \frac{1}{12}n\sigma_3(\frac{n}{2}) + \left(\frac{1}{48}n - \frac{1}{12}n^2\right)\sigma_1(\frac{n}{2}) \tag{1.16}
$$

in ([11], Theorem 4.1) and

Proposition 1.3. *(See ([12], Theorem 1.1)) Let $n \in \mathbb{N}$. Then we have*

(a)

$$T_{m,1,3}(n) = \frac{1}{1440}\left[n\left\{5\sigma_5(n) + 16\sigma_5(\frac{n}{2}) - 9n\sigma_3(n) - 3\sigma_1(\frac{n}{2})\right\} + 4b(n)\right],$$
$$T_{m,3,1}(n) = \frac{1}{720}\left[n\left\{\sigma_5(n) + 20\sigma_5(\frac{n}{2}) - (36n-15)\,\sigma_3(\frac{n}{2})\right\} - b(n)\right],$$

(b)

$$\begin{aligned} T_{m,1,5}(n) &= \frac{1}{17136}\left\{21n\sigma_7(n) + 64n\sigma_7(\frac{n}{2}) - 51n^2\sigma_5(n) + 17n\sigma_1(\frac{n}{2}) + 1632d(n)\right. \\ &\quad \left. + 51c(n) - 21nb(n)\right\}, \\ T_{m,3,3}(n) &= \frac{1}{16320}\left\{2n\sigma_7(n) + 32n\sigma_7(\frac{n}{2}) - 34n\sigma_3(\frac{n}{2}) - 544d(n) - 17c(n)\right. \\ &\quad \left. + 15nb(n)\right\}, \\ T_{m,5,1}(n) &= \frac{1}{22848}\left\{4n\sigma_7(n) + 336n\sigma_7(\frac{n}{2}) - 816n^2\sigma_5(\frac{n}{2}) + 476n\sigma_5(\frac{n}{2}) + 544d(n)\right. \\ &\quad \left. + 17c(n) - 21nb(n)\right\}, \end{aligned}$$

(c)

$$\begin{aligned} T_{m,1,7}(n) &= \frac{1}{446400}\left\{255n\sigma_9(n) + 768n\sigma_9(\frac{n}{2}) - 775n^2\sigma_7(n) - 465n\sigma_1(\frac{n}{2})\right. \\ &\quad \left. + 1240\tau(n) + 190464\tau(\frac{n}{2}) - 23040nd(n) - 720nc(n)\right\}, \\ T_{m,3,5}(n) &= \frac{1}{781200}\left\{21n\sigma_9(n) + 320n\sigma_9(\frac{n}{2}) + 775n\sigma_3(\frac{n}{2}) - 651\tau(n) - 59520\tau(\frac{n}{2})\right. \\ &\quad \left. + 20160nd(n) + 630nc(n)\right\}, \\ T_{m,5,3}(n) &= \frac{1}{520800}\left\{5n\sigma_9(n) + 336n\sigma_9(\frac{n}{2}) - 1085n\sigma_5(\frac{n}{2}) + 310\tau(n) + 13888\tau(\frac{n}{2})\right. \\ &\quad \left. - 10080nd(n) - 315nc(n)\right\}, \\ T_{m,7,1}(n) &= \frac{1}{111600}\left\{3n\sigma_9(n) + 1020n\sigma_9(\frac{n}{2}) - 3100n^2\sigma_7(\frac{n}{2}) + 2325n\sigma_7(\frac{n}{2})\right. \\ &\quad \left. - 93\tau(n) - 2480\tau(\frac{n}{2}) + 2880nd(n) + 90nc(n)\right\}. \end{aligned}$$

In this paper we desire to expand Proposition 1.3 into $T^2_{m,e,f}(n)$ and so we obtain the following theorem :

Theorem 1.1. *Let* $n \in \mathbb{N}$. *Then we have*

(a)

$$\begin{aligned} T^2_{m,1,1}(n) &= \sum_{m<\frac{n}{2}} m^2\sigma_1(m)\sigma_1(n-2m) \\ &= \frac{1}{960}\left[n^2\left\{6\sigma_3(n) + 24\sigma_3(\frac{n}{2}) - 5n\sigma_1(n) - 10\,(3n-1)\,\sigma_1(\frac{n}{2})\right\} - b(n)\right], \end{aligned}$$

(b)

$$\begin{aligned} T^2_{m,1,3}(n) &= \sum_{m<\frac{n}{2}} m^2\sigma_1(m)\sigma_3(n-2m) \\ &= \frac{1}{6720}\left\{5n^2\sigma_5(n) + 16n^2\sigma_5(\frac{n}{2}) - 7n^3\sigma_3(n) - 7n^2\sigma_1(\frac{n}{2}) - 160d(n)\right. \\ &\quad \left. - 5c(n) + 7nb(n)\right\}, \end{aligned}$$

(c)

$$T^2_{m,3,1}(n) = \sum_{m<\frac{n}{2}} m^2\sigma_3(m)\sigma_1(n-2m)$$
$$= \frac{1}{8064}\left\{4n^2\sigma_5(n) + 80n^2\sigma_5(\frac{n}{2}) - 168n^3\sigma_3(\frac{n}{2}) + 84n^2\sigma_3(\frac{n}{2}) + 96d(n)\right.$$
$$\left. +3c(n) - 7nb(n)\right\},$$

(d)

$$T^2_{m,1,5}(n) = \sum_{m<\frac{n}{2}} m^2\sigma_1(m)\sigma_5(n-2m)$$
$$= \frac{1}{2056320}\left\{420n^2\sigma_7(n) + 1280n^2\sigma_7(\frac{n}{2}) - 765n^3\sigma_5(n) + 1020n^2\sigma_1(\frac{n}{2})\right.$$
$$\left. +58752nd(n) + 1836nc(n) - 420n^2b(n) - 1071\tau(n) - 121856\tau(\frac{n}{2})\right\},$$

(e)

$$T^2_{m,3,3}(n) = \sum_{m<\frac{n}{2}} m^2\sigma_3(m)\sigma_3(n-2m)$$
$$= \frac{1}{293760}\left\{10n^2\sigma_7(n) + 160n^2\sigma_7(\frac{n}{2}) - 306n^2\sigma_3(\frac{n}{2}) - 4896nd(n)\right.$$
$$\left. -153nc(n) + 75n^2b(n) + 68\tau(n) + 4352\tau(\frac{n}{2})\right\},$$

(f)

$$T^2_{m,5,1}(n) = \sum_{m<\frac{n}{2}} m^2\sigma_5(m)\sigma_1(n-2m)$$
$$= \frac{1}{293760}\left\{20n^2\sigma_7(n) + 1680n^2\sigma_7(\frac{n}{2}) - 1530n^2\left(3n-2\right)\sigma_5(\frac{n}{2}) + 4896nd(n)\right.$$
$$\left. +153nc(n) - 105n^2b(n) - 68\tau(n) - 2448\tau(\frac{n}{2})\right\}.$$

In addition, using Theorem 1.1 we can easily obtain some identities induced from $T^2_{m,e,f}(n)$, which are introduced in the following theorem :

Theorem 1.2. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$L^3(q)L(q^2)$$
$$= 1 + \frac{2016}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^n + \frac{12288}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n} - 1244\sum_{n=1}^{\infty}n\sigma_5(n)q^n$$
$$- 4608\sum_{n=1}^{\infty}n\sigma_5(n)q^{2n} + \frac{18144}{5}\sum_{n=1}^{\infty}n^2\sigma_3(n)q^n + \frac{82944}{5}\sum_{n=1}^{\infty}n^2\sigma_3(n)q^{2n}$$
$$- 2592\sum_{n=1}^{\infty}n^3\sigma_1(n)q^n - 13824\sum_{n=1}^{\infty}n^3\sigma_1(n)q^{2n} - \frac{288}{85}\sum_{n=1}^{\infty}b(n)q^n,$$

(b)

$$\begin{aligned}
&L^3(q^2)M(q)\\
&= 1 - \frac{120}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^n - \frac{8064}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^{2n} + \frac{1080}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^n\\
&\quad + \frac{34560}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^{2n} - \frac{2160}{7}\sum_{n=1}^{\infty}n^2\sigma_5(n)q^n - \frac{27648}{7}\sum_{n=1}^{\infty}n^2\sigma_5(n)q^{2n}\\
&\quad + 432\sum_{n=1}^{\infty}n^3\sigma_3(n)q^n + \frac{86400}{217}\sum_{n=1}^{\infty}d(n)q^n + \frac{2700}{217}\sum_{n=1}^{\infty}c(n)q^n\\
&\quad + \frac{756}{17}\sum_{n=1}^{\infty}nb(n)q^n,
\end{aligned}$$

(c)

$$\begin{aligned}
&L^3(q)M(q^2)\\
&= 1 - \frac{504}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^n - \frac{7680}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^{2n} + \frac{2160}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^n\\
&\quad + \frac{69120}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^{2n} - \frac{1728}{7}\sum_{n=1}^{\infty}n^2\sigma_5(n)q^n - \frac{138240}{7}\sum_{n=1}^{\infty}n^2\sigma_5(n)q^{2n}\\
&\quad + 27648\sum_{n=1}^{\infty}n^3\sigma_3(n)q^{2n} - \frac{172800}{217}\sum_{n=1}^{\infty}d(n)q^n - \frac{5400}{217}\sum_{n=1}^{\infty}c(n)q^n\\
&\quad + \frac{1512}{17}\sum_{n=1}^{\infty}nb(n)q^n,
\end{aligned}$$

(d)

$$\begin{aligned}
&L(q)L^2(q^2)N(q)\\
&= 1 + \frac{4080}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{61440}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} - \frac{15264}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^n\\
&\quad - \frac{165888}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} + \frac{8112}{17}\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n + \frac{49152}{17}\sum_{n=1}^{\infty}n^2\sigma_7(n)q^{2n}\\
&\quad - 648\sum_{n=1}^{\infty}n^3\sigma_5(n)q^n - \frac{82944}{155}\sum_{n=1}^{\infty}nd(n)q^n - \frac{2592}{155}\sum_{n=1}^{\infty}nc(n)q^n\\
&\quad - \frac{4032}{17}\sum_{n=1}^{\infty}n^2b(n)q^n - \frac{36984}{3455}\sum_{n=1}^{\infty}\tau(n)q^n - \frac{61440}{691}\sum_{n=1}^{\infty}\tau(n)q^{2n},
\end{aligned}$$

(e)

$$\begin{aligned}
&L^2(q^2)M(q)M(q^2)\\
&= 1 + \frac{240}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{65280}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} - \frac{144}{31}\sum_{n=1}^{\infty}n\sigma_9(n)q^n\\
&\quad - \frac{96768}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} + \frac{240}{17}\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n + \frac{15360}{17}\sum_{n=1}^{\infty}n^2\sigma_7(n)q^{2n}\\
&\quad + \frac{76032}{31}\sum_{n=1}^{\infty}nd(n)q^n + \frac{2376}{31}\sum_{n=1}^{\infty}nc(n)q^n + \frac{1800}{17}\sum_{n=1}^{\infty}n^2b(n)q^n\\
&\quad + \frac{32928}{691}\sum_{n=1}^{\infty}\tau(n)q^n + \frac{2857728}{3455}\sum_{n=1}^{\infty}\tau(n)q^{2n},
\end{aligned}$$

(f)

$$\begin{aligned}
&L(q)L^2(q^2)N(q^2)\\
&= 1 + \frac{48}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{65472}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} - \frac{144}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^n\\
&\quad - \frac{196128}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} + \frac{48}{17}\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n + \frac{81408}{17}\sum_{n=1}^{\infty}n^2\sigma_7(n)q^{2n}\\
&\quad - 5184\sum_{n=1}^{\infty}n^3\sigma_5(n)q^{2n} - \frac{31104}{155}\sum_{n=1}^{\infty}nd(n)q^n - \frac{972}{155}\sum_{n=1}^{\infty}nc(n)q^n\\
&\quad - \frac{252}{17}\sum_{n=1}^{\infty}n^2b(n)q^n - \frac{16824}{3455}\sum_{n=1}^{\infty}\tau(n)q^n - \frac{65472}{691}\sum_{n=1}^{\infty}\tau(n)q^{2n}.
\end{aligned}$$

Finally, we can deduce Theorem 1.3 :

Theorem 1.3. *Let* $n \in \mathbb{N}$. *Then we have*

(a)

$$\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_{1,1}(a;2)\sigma_{3,1}(b;2)\sigma_1(c) = \sum_{m=1}^{n-1}\sum_{k=1}^{m-1}k\sigma_{1,1}(k;2)\sigma_{3,1}(m-k;2)\sigma_1(n-m)\\
&= \frac{1}{685440}\Big\{133n\sigma_7(n) + 4032n\sigma_7(\frac{n}{2}) - 17n\,(12n-7)\,\sigma_5(n) - 816n\,(12n-7)\,\sigma_5(\frac{n}{2})\\
&\quad + 2499n\sigma_3(n) - 6664n\sigma_3(\frac{n}{2}) - 833n\,(4n-1)\,\sigma_1(n) + 1666n\,(4n-1)\,\sigma_1(\frac{n}{2})\\
&\quad -8704d(n) - 272c(n) + 56\,(21n-17)\,b(n)\big\},
\end{aligned}$$

(b)

$$\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_{1,1}(a;2)\sigma_{3,0}(b;2)\sigma_1(c)\\
&= \frac{1}{514080}\Big\{126n\sigma_7(n) - 3696n\sigma_7(\frac{n}{2}) - 34n\,(12n-7)\,\sigma_5(n) + 748n\,(12n-7)\,\sigma_5(\frac{n}{2})\\
&\quad - 2142n\sigma_3(n) + 5712n\sigma_3(\frac{n}{2}) + 714n\,(4n-1)\,\sigma_1(n) - 1428n\,(4n-1)\,\sigma_1(\frac{n}{2})\\
&\quad +1632d(n) + 51c(n) - 7\,(69n-68)\,b(n)\big\},
\end{aligned}$$

(c)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_{1,0}(a;2)\sigma_{3,0}(b;2)\sigma_1(c)$$
$$= \frac{1}{42840}\Big\{7n\sigma_7(n) + 588n\sigma_7(\frac{n}{2}) - 17n^2\sigma_5(n) - 17n\,(104n-49)\,\sigma_5(\frac{n}{2})$$
$$- 119n\sigma_3(n) + 119n\left(6n^2-3n-4\right)\sigma_3(\frac{n}{2}) + 119n^2\sigma_1(n)$$
$$+119n\,(4n-1)\,\sigma_1(\frac{n}{2}) + 544d(n) + 17c(n) - 7nb(n)\Big\},$$

(d)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_{1,0}(a;2)\sigma_{3,1}(b;2)\sigma_1(c)$$
$$= \frac{1}{1028160}\Big\{273n\sigma_7(n) - 12768n\sigma_7(\frac{n}{2}) - 17n\,(99n-35)\,\sigma_5(n)$$
$$+ 136n\,(288n-133)\,\sigma_5(\frac{n}{2}) + 357n\left(6n^2-3n+7\right)\sigma_3(n)$$
$$- 1428n\left(12n^2-6n-7\right)\sigma_3(\frac{n}{2}) - 2499n^2\sigma_1(n) - 2499n\,(4n-1)\,\sigma_1(\frac{n}{2})$$
$$-3264d(n) - 102c(n) - 14\,(45n-34)\,b(n)\}.$$

Remark 1.1. Theorem 1.1 and Theorem 1.2 help us to approach easily to another convolution sums, for example, $T^3_{m,e,f}(n)$, $T^4_{m,e,f}(n)$, etc., and we expect that this work allows us to find some relations between divisor functions and the infinite product sums.

2 The Various Convolution Sums Multiplying m^2

We introduce Proposition 2.1 to obtain some various convolution sum formulae.

Proposition 2.1. *(See ([11], Theorem 1.1)) For $q\in\mathbb{C}$ with $|q|<1$, we have*

(a)

$$L(q)M(q^2) = 1 - 24\sum_{n=1}^{\infty}\sigma_5(n)q^n + 1440\sum_{n=1}^{\infty}n\sigma_3(n)q^{2n} - 480\sum_{n=1}^{\infty}\sigma_5(n)q^{2n},$$

(b)

$$L(q^2)M(q) = 1 + 360\sum_{n=1}^{\infty}n\sigma_3(n)q^n - 120\sum_{n=1}^{\infty}\sigma_5(n)q^n - 384\sum_{n=1}^{\infty}\sigma_5(n)q^{2n},$$

(c)

$$L(q)L^2(q^2) = 1 - 144\sum_{n=1}^{\infty}n^2\sigma_1(n)q^n + 144\sum_{n=1}^{\infty}n\sigma_3(n)q^n - 24\sum_{n=1}^{\infty}\sigma_5(n)q^n$$
$$- 2304\sum_{n=1}^{\infty}n^2\sigma_1(n)q^{2n} + 2592\sum_{n=1}^{\infty}n\sigma_3(n)q^{2n} - 480\sum_{n=1}^{\infty}\sigma_5(n)q^{2n},$$

(d)

$$L^2(q)L(q^2) = 1 - 576\sum_{n=1}^{\infty} n^2\sigma_1(n)q^n + 648\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120\sum_{n=1}^{\infty}\sigma_5(n)q^n \\ - 2304\sum_{n=1}^{\infty} n^2\sigma_1(n)q^{2n} + 2304\sum_{n=1}^{\infty} n\sigma_3(n)q^{2n} - 384\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}.$$

Proof of Theorem 1.1. (a) By (1.5) and (1.9), we have

$$\begin{aligned} L^3(q)L(q^2) &= L^3(q)\cdot L(q^2) \\ &= \left(1 - 1728\sum_{n=1}^{\infty} n^2\sigma_1(n)q^n + 2160\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right) \\ &\quad \times \left(1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^{2m}\right) \\ &= 1 + \sum_{N=1}^{\infty}\Big\{-1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) - 24\sigma_1(\frac{N}{2}) \\ &\quad + 1728\cdot 24\sum_{m<\frac{N}{2}}(N-2m)^2\,\sigma_1(N-2m)\sigma_1(m) \\ &\quad - 2160\cdot 24\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_3(N-2m)\sigma_1(m) \\ &\quad +504\cdot 24\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\sigma_1(m)\Big\}q^N \\ &= 1 + \sum_{N=1}^{\infty}\Big\{-1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) - 24\sigma_1(\frac{N}{2}) \\ &\quad + 1728\cdot 24N^2\cdot T_{1,2}(N) - 1728\cdot 24\cdot 4N\cdot T_{m,1,1}(N) \\ &\quad + 1728\cdot 24\cdot 4\sum_{m<\frac{N}{2}} m^2\sigma_1(m)\sigma_1(N-2m) - 2160\cdot 24N\cdot T_{1,3}(N) \\ &\quad +2160\cdot 24\cdot 2\cdot T_{m,1,3}(N) + 504\cdot 24\cdot T_{1,5}(N)\Big\}q^N. \end{aligned} \tag{2.1}$$

Also, by (1.5) and Proposition 2.1 (d), we note that

$$
\begin{aligned}
&L^3(q)L(q^2) = L(q)\cdot L^2(q)L(q^2)\\
&= \left(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\left(1-576\sum_{m=1}^{\infty}m^2\sigma_1(m)q^m+648\sum_{m=1}^{\infty}m\sigma_3(m)q^m\right.\\
&\quad -120\sum_{m=1}^{\infty}\sigma_5(m)q^m-2304\sum_{m=1}^{\infty}m^2\sigma_1(m)q^{2m}+2304\sum_{m=1}^{\infty}m\sigma_3(m)q^{2m}\\
&\quad \left.-384\sum_{m=1}^{\infty}\sigma_5(m)q^{2m}\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{-576N^2\sigma_1(N)+648N\sigma_3(N)-120\sigma_5(N)-576N^2\sigma_1(\frac{N}{2})\right.\\
&\quad +1152N\sigma_3(\frac{N}{2})-384\sigma_5(\frac{N}{2})-24\sigma_1(N)+24\cdot 576\sum_{m=1}^{N-1}\sigma_1(N-m)\cdot m^2\sigma_1(m)\\
&\quad -24\cdot 648\sum_{m=1}^{N-1}\sigma_1(N-m)\cdot m\sigma_3(m)+24\cdot 120\sum_{m=1}^{N-1}\sigma_1(N-m)\sigma_5(m)\\
&\quad +24\cdot 2304\sum_{m<\frac{N}{2}}\sigma_1(N-2m)\cdot m^2\sigma_1(m)\\
&\quad \left.-24\cdot 2304\sum_{m<\frac{N}{2}}\sigma_1(N-2m)\cdot m\sigma_3(m)+24\cdot 384\sum_{m<\frac{N}{2}}\sigma_1(N-2m)\sigma_5(m)\right\}q^N\\
&= 1+\sum_{N=1}^{\infty}\left\{-576N^2\sigma_1(N)+648N\sigma_3(N)-120\sigma_5(N)-576N^2\sigma_1(\frac{N}{2})\right.\\
&\quad +1152N\sigma_3(\frac{N}{2})-384\sigma_5(\frac{N}{2})-24\sigma_1(N)+24\cdot 576\cdot I^2_{m,1,1}(N)\\
&\quad -24\cdot 648\cdot I_{m,3,1}(N)+24\cdot 120\cdot I_{1,5}(N)\\
&\quad +24\cdot 2304\sum_{m<\frac{N}{2}}m^2\sigma_1(m)\sigma_1(N-2m)-24\cdot 2304\cdot T_{m,3,1}(N)\\
&\quad \left.+24\cdot 384\cdot T_{5,1}(N)\right\}q^N.
\end{aligned}
\tag{2.2}
$$

Therefore, equating (2.1) with (2.2) and using Proposition 1.1 (b), Proposition 1.2 (a), (1.16), and Proposition 1.3 (a), we obtain the proof.

(b) By (1.6) and (1.9), we have

$$\begin{aligned}
&L^3(q^2)M(q) = M(q)\cdot L^3(q^2)\\
&= \left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\\
&\quad\times\left(1 - 1728\sum_{m=1}^{\infty}m^2\sigma_1(m)q^{2m} + 2160\sum_{n=1}^{\infty}m\sigma_3(m)q^{2m} - 504\sum_{m=1}^{\infty}\sigma_5(m)q^{2m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\left\{-432N^2\sigma_1(\frac{N}{2}) + 1080N\sigma_3(\frac{N}{2}) - 504\sigma_5(\frac{N}{2}) + 240\sigma_3(N)\right.\\
&\quad - 240\cdot 1728\sum_{m<\frac{N}{2}}\sigma_3(N-2m)\cdot m^2\sigma_1(m)\\
&\quad + 240\cdot 2160\sum_{m<\frac{N}{2}}\sigma_3(N-2m)\cdot m\sigma_3(m)\\
&\quad \left.-240\cdot 504\sum_{m<\frac{N}{2}}\sigma_3(N-2m)\sigma_5(m)\right\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\left\{-432N^2\sigma_1(\frac{N}{2}) + 1080N\sigma_3(\frac{N}{2}) - 504\sigma_5(\frac{N}{2}) + 240\sigma_3(N)\right.\\
&\quad - 240\cdot 1728\sum_{m<\frac{N}{2}}m^2\sigma_1(m)\sigma_3(N-2m) + 240\cdot 2160\cdot T_{m,3,3}(N)\\
&\quad \left.-240\cdot 504\cdot T_{5,3}(N)\right\}q^N.
\end{aligned} \tag{2.3}$$

And by (1.8) and Proposition 2.1 (b), we obtain

$$
\begin{aligned}
&L^3(q^2)M(q) = L(q^2)M(q)\cdot L^2(q^2)\\
&= \left(1 + 360\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120\sum_{n=1}^{\infty}\sigma_5(n)q^n - 384\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\right)\\
&\quad\times\left(1 - 288\sum_{m=1}^{\infty} m\sigma_1(m)q^{2m} + 240\sum_{m=1}^{\infty}\sigma_3(m)q^{2m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\Big\{-144N\sigma_1(\frac{N}{2}) + 240\sigma_3(\frac{N}{2}) + 360N\sigma_3(N)\\
&\quad - 360\cdot 288\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_3(N-2m)\cdot m\sigma_1(m)\\
&\quad + 360\cdot 240\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_3(N-2m)\sigma_3(m) - 120\sigma_5(N)\\
&\quad + 120\cdot 288\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\cdot m\sigma_1(m) - 120\cdot 240\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\sigma_3(m)\\
&\quad - 384\sigma_5(\frac{N}{2}) + 384\cdot 288\sum_{m<\frac{N}{2}}\sigma_5(\frac{N}{2}-m)\cdot m\sigma_1(m)\\
&\quad -384\cdot 240\sum_{m<\frac{N}{2}}\sigma_5(\frac{N}{2}-m)\sigma_3(m)\Big\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\Big\{-144N\sigma_1(\frac{N}{2}) + 240\sigma_3(\frac{N}{2}) + 360N\sigma_3(N) - 360\cdot 288N\cdot T_{m,1,3}(N)\\
&\quad + 360\cdot 288\cdot 2\sum_{m<\frac{N}{2}} m^2\sigma_1(m)\sigma_3(N-2m) + 360\cdot 240N\cdot T_{3,3}(N)\\
&\quad - 360\cdot 240\cdot 2\cdot T_{m,3,3}(N) - 120\sigma_5(N) + 120\cdot 288\cdot T_{m,1,5}(N)\\
&\quad - 120\cdot 240\cdot T_{3,5}(N) - 384\sigma_5(\frac{N}{2}) + 384\cdot 288\cdot I_{m,1,5}(\frac{N}{2})\\
&\quad -384\cdot 240\cdot I_{3,5}(\frac{N}{2})\Big\}q^N.
\end{aligned}
\tag{2.4}
$$

So we equate (2.3) with (2.4) and refer to Proposition 1.1 (c), Proposition 1.3 (a) and (b).

(c) From (1.6) and (1.9) we note that

$$
\begin{aligned}
&L^3(q)M(q^2) = L^3(q) \cdot M(q^2) \\
&= \left(1 - 1728\sum_{n=1}^{\infty} n^2\sigma_1(n)q^n + 2160\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right) \\
&\quad \times \left(1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^{2m}\right) \\
&= 1 + \sum_{N=1}^{\infty}\left\{-1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) + 240\sigma_3(\frac{N}{2})\right. \\
&\quad - 1728\cdot 240\sum_{m<\frac{N}{2}}(N-2m)^2\,\sigma_1(N-2m)\sigma_3(m) \\
&\quad + 2160\cdot 240\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_3(N-2m)\sigma_3(m) \\
&\quad \left. -504\cdot 240\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\sigma_3(m)\right\}q^N \\
&= 1 + \sum_{N=1}^{\infty}\left\{-1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) + 240\sigma_3(\frac{N}{2})\right. \\
&\quad - 1728\cdot 240N^2\cdot T_{3,1}(N) + 1728\cdot 240\cdot 4N\cdot T_{m,3,1}(N) \\
&\quad - 1728\cdot 240\cdot 4\sum_{m<\frac{N}{2}} m^2\sigma_3(m)\sigma_1(N-2m) + 2160\cdot 240N\cdot T_{3,3}(N) \\
&\quad \left. -2160\cdot 240\cdot 2\cdot T_{m,3,3}(N) - 504\cdot 240\cdot T_{3,5}(N)\right\}q^N.
\end{aligned}
\tag{2.5}
$$

Similarly, by (1.8) and Proposition 2.1 (a), we have

$$
\begin{aligned}
&L^3(q)M(q^2) = L^2(q)\cdot L(q)M(q^2)\\
&= \left(1 - 288\sum_{n=1}^{\infty} n\sigma_1(n)q^n + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\\
&\quad\times\left(1 - 24\sum_{m=1}^{\infty}\sigma_5(m)q^m + 1440\sum_{m=1}^{\infty} m\sigma_3(m)q^{2m} - 480\sum_{m=1}^{\infty}\sigma_5(m)q^{2m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\Bigg\{-24\sigma_5(N) + 720N\sigma_3(\frac{N}{2}) - 480\sigma_5(\frac{N}{2}) - 288N\sigma_1(N)\\
&\quad + 288\cdot 24\sum_{m=1}^{N-1}(N-m)\,\sigma_1(N-m)\sigma_5(m)\\
&\quad - 288\cdot 1440\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_1(N-2m)\cdot m\sigma_3(m)\\
&\quad + 288\cdot 480\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_1(N-2m)\sigma_5(m) + 240\sigma_3(N)\\
&\quad - 240\cdot 24\sum_{m=1}^{n-1}\sigma_3(N-m)\sigma_5(m) + 240\cdot 1440\sum_{m<\frac{N}{2}}\sigma_3(N-2m)\cdot m\sigma_3(m)\\
&\quad -240\cdot 480\sum_{m<\frac{N}{2}}\sigma_3(N-2m)\sigma_5(m)\Bigg\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\Big\{-24\sigma_5(N) + 720N\sigma_3(\frac{N}{2}) - 480\sigma_5(\frac{N}{2}) - 288N\sigma_1(N) + 240\sigma_3(N)\\
&\quad + 288\cdot 24N\cdot I_{1,5}(N) - 288\cdot 24\cdot I_{m,5,1}(N) - 288\cdot 1440N\cdot T_{m,3,1}(N)\\
&\quad + 288\cdot 1440\cdot 2\sum_{m<\frac{N}{2}} m^2\sigma_3(m)\sigma_1(N-2m) + 288\cdot 480N\cdot T_{5,1}(N)\\
&\quad - 288\cdot 480\cdot 2\cdot T_{m,5,1}(N) - 240\cdot 24\cdot I_{3,5}(N) + 240\cdot 1440\cdot T_{m,3,3}(N)\\
&\quad -240\cdot 480\cdot T_{5,3}(N)\big\}\,q^N.
\end{aligned}
\tag{2.6}
$$

Therefore, equating (2.5) with (2.6) and appealing to Proposition 1.1 (c), Proposition 1.3 (a) and (b), we obtain the proof.

(d) By (1.8) and (1.12), we obtain

$$
\begin{aligned}
&L(q)L^2(q^2)N(q) = L(q)N(q)\cdot L^2(q^2) \\
&= \left(1 - 1008\sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right) \\
&\quad\times\left(1 - 288\sum_{m=1}^{\infty} m\sigma_1(m)q^{2m} + 240\sum_{m=1}^{\infty}\sigma_3(m)q^{2m}\right) \\
&= 1 + \sum_{N=1}^{\infty}\left\{-144N\sigma_1(\frac{N}{2}) + 240\sigma_3(\frac{N}{2}) - 1008N\sigma_5(N)\right. \\
&\quad + 1008\cdot 288\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_5(N-2m)\cdot m\sigma_1(m) \\
&\quad - 1008\cdot 240\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_5(N-2m)\sigma_3(m) + 480\sigma_7(N) \\
&\quad - 480\cdot 288\sum_{m<\frac{N}{2}}\sigma_7(N-2m)\cdot m\sigma_1(m) \\
&\quad \left.+480\cdot 240\sum_{m<\frac{N}{2}}\sigma_7(N-2m)\sigma_3(m)\right\} q^N \\
&= 1 + \sum_{N=1}^{\infty}\left\{-144N\sigma_1(\frac{N}{2}) + 240\sigma_3(\frac{N}{2}) - 1008N\sigma_5(N) + 1008\cdot 288N\cdot T_{m,1,5}(N)\right. \\
&\quad - 1008\cdot 288\cdot 2\sum_{m<\frac{N}{2}} m^2\sigma_1(m)\sigma_5(N-2m) - 1008\cdot 240N\cdot T_{3,5}(N) \\
&\quad + 1008\cdot 240\cdot 2\cdot T_{m,3,5}(N) + 480\sigma_7(N) - 480\cdot 288\cdot T_{m,1,7}(N) \\
&\quad +480\cdot 240\cdot T_{3,7}(N)\}\, q^N.
\end{aligned}
\tag{2.7}
$$

Also by (1.7) and Proposition 2.1 (c), we observe that

$$
\begin{aligned}
&L(q)L^2(q^2)N(q) = N(q)\cdot L(q)L^2(q^2)\\
&= \left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)\left(1-144\sum_{m=1}^{\infty}m^2\sigma_1(m)q^m+144\sum_{m=1}^{\infty}m\sigma_3(m)q^m\right.\\
&\quad -24\sum_{m=1}^{\infty}\sigma_5(m)q^m-2304\sum_{m=1}^{\infty}m^2\sigma_1(m)q^{2m}+2592\sum_{m=1}^{\infty}m\sigma_3(m)q^{2m}\\
&\quad \left.-480\sum_{m=1}^{\infty}\sigma_5(m)q^{2m}\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{-144N^2\sigma_1(N)+144N\sigma_3(N)-24\sigma_5(N)-576N^2\sigma_1(\frac{N}{2})\right.\\
&\quad +1296N\sigma_3(\frac{N}{2})-480\sigma_5(\frac{N}{2})-504\sigma_5(N)\\
&\quad +504\cdot144\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m^2\sigma_1(m)\\
&\quad -504\cdot144\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m\sigma_3(m)+504\cdot24\sum_{m=1}^{N-1}\sigma_5(N-m)\sigma_5(m) \qquad (2.8)\\
&\quad +504\cdot2304\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\cdot m^2\sigma_1(m)\\
&\quad -504\cdot2592\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\cdot m\sigma_3(m)\\
&\quad \left.+504\cdot480\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\sigma_5(m)\right\}q^N\\
&= 1+\sum_{N=1}^{\infty}\left\{-144N^2\sigma_1(N)+144N\sigma_3(N)-528\sigma_5(N)-576N^2\sigma_1(\frac{N}{2})\right.\\
&\quad +1296N\sigma_3(\frac{N}{2})-480\sigma_5(\frac{N}{2})+504\cdot144\cdot I^2_{m,1,5}(N)-504\cdot144\cdot I_{m,3,5}(N)\\
&\quad +504\cdot24\cdot I_{5,5}(N)+504\cdot2304\sum_{m<\frac{N}{2}}m^2\sigma_1(m)\sigma_5(N-2m)\\
&\quad -504\cdot2592\cdot T_{m,3,5}(N)+504\cdot480\cdot T_{5,5}(N)\}\,q^N.
\end{aligned}
$$

Thus we equate (2.7) with (2.8) and use Proposition 1.1 (d), Proposition 1.2 (c), Proposition 1.3 (b) and (c).

(e) From (1.6) and (1.11) we have

$$
\begin{aligned}
&L^2(q^2)M(q)M(q^2) = M(q) \cdot L^2(q^2)M(q^2) \\
&= \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right) \\
&\quad \times \left(1 + 1728 \sum_{m=1}^{\infty} m^2 \sigma_3(m) q^{2m} - 2016 \sum_{m=1}^{\infty} m\sigma_5(m) q^{2m} + 480 \sum_{m=1}^{\infty} \sigma_7(m) q^{2m}\right) \\
&= 1 + \sum_{N=1}^{\infty} \left\{ 432N^2\sigma_3(\frac{N}{2}) - 1008N\sigma_5(\frac{N}{2}) + 480\sigma_7(\frac{N}{2}) + 240\sigma_3(N) \right. \\
&\quad + 240 \cdot 1728 \sum_{m<\frac{N}{2}} \sigma_3(N-2m) \cdot m^2\sigma_3(m) \\
&\quad - 240 \cdot 2016 \sum_{m<\frac{N}{2}} \sigma_3(N-2m) \cdot m\sigma_5(m) \\
&\quad \left. +240 \cdot 480 \sum_{m<\frac{N}{2}} \sigma_3(N-2m)\sigma_7(m) \right\} q^N \\
&= 1 + \sum_{N=1}^{\infty} \left\{ 432N^2\sigma_3(\frac{N}{2}) - 1008N\sigma_5(\frac{N}{2}) + 480\sigma_7(\frac{N}{2}) + 240\sigma_3(N) \right. \\
&\quad + 240 \cdot 1728 \sum_{m<\frac{N}{2}} m^2\sigma_3(m)\sigma_3(N-2m) - 240 \cdot 2016 \cdot T_{m,5,3}(N) \\
&\quad \left. +240 \cdot 480 \cdot T_{7,3}(N) \right\} q^N.
\end{aligned}
\tag{2.9}
$$

Similarly, by (1.10) and Proposition 2.1 (b), we obtain

$$
\begin{aligned}
&L^2(q^2)M(q)M(q^2) = L(q^2)M(q)\cdot L(q^2)M(q^2)\\
&= \left(1+360\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120\sum_{n=1}^{\infty}\sigma_5(n)q^n - 384\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\right)\\
&\quad\times\left(1+720\sum_{m=1}^{\infty} m\sigma_3(m)q^{2m} - 504\sum_{m=1}^{\infty}\sigma_5(m)q^{2m}\right)\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{360N\sigma_3(\frac{N}{2}) - 504\sigma_5(\frac{N}{2}) + 360N\sigma_3(N)\\
&\quad + 360\cdot 720\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_3(N-2m)\cdot m\sigma_3(m)\\
&\quad - 360\cdot 504\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_3(N-2m)\sigma_5(m) - 120\sigma_5(N)\\
&\quad - 120\cdot 720\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\cdot m\sigma_3(m)\\
&\quad + 120\cdot 504\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\sigma_5(m) - 384\sigma_5(\frac{N}{2})\\
&\quad -384\cdot 720\sum_{m<\frac{N}{2}}\sigma_5(\frac{N}{2}-m)\cdot m\sigma_3(m) + 384\cdot 504\sum_{m<\frac{N}{2}}\sigma_5(\frac{N}{2}-m)\sigma_5(m)\Bigg\}q^N\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{360N\sigma_3(\frac{N}{2}) - 888\sigma_5(\frac{N}{2}) + 360N\sigma_3(N) - 120\sigma_5(N)\\
&\quad + 360\cdot 720N\cdot T_{m,3,3}(N) - 360\cdot 720\cdot 2\sum_{m<\frac{N}{2}} m^2\sigma_3(m)\sigma_3(N-2m)\\
&\quad - 360\cdot 504N\cdot T_{5,3}(N) + 360\cdot 504\cdot 2\cdot T_{m,5,3}(N) - 120\cdot 720\cdot T_{m,3,5}(N)\\
&\quad +120\cdot 504\cdot T_{5,5}(N) - 384\cdot 720\cdot I_{m,3,5}(\frac{N}{2}) + 384\cdot 504\cdot I_{5,5}(\frac{N}{2})\Bigg\}q^N.
\end{aligned}
\tag{2.10}
$$

So equating (2.9) with (2.10) and using Proposition 1.1 (d), Proposition 1.3 (b) and (c), we can complete the proof.

(f) By (1.5) and (1.13), we note that

$$
\begin{aligned}
&L(q)L^2(q^2)N(q^2) = L(q)\cdot L^2(q^2)N(q^2)\\
&= \left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\\
&\quad\times\left(1 - 1728\sum_{m=1}^{\infty}m^2\sigma_5(m)q^{2m} + 1440\sum_{m=1}^{\infty}m\sigma_7(m)q^{2m} - 264\sum_{m=1}^{\infty}\sigma_9(m)q^{2m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\left\{-432N^2\sigma_5(\frac{N}{2}) + 720N\sigma_7(\frac{N}{2}) - 264\sigma_9(\frac{N}{2}) - 24\sigma_1(N)\right.\\
&\quad + 24\cdot 1728\sum_{m<\frac{N}{2}}\sigma_1(N-2m)\cdot m^2\sigma_5(m) - 24\cdot 1440\sum_{m<\frac{N}{2}}\sigma_1(N-2m)\cdot m\sigma_7(m)\\
&\quad\left. +24\cdot 264\sum_{m<\frac{N}{2}}\sigma_1(N-2m)\sigma_9(m)\right\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\left\{-432N^2\sigma_5(\frac{N}{2}) + 720N\sigma_7(\frac{N}{2}) - 264\sigma_9(\frac{N}{2}) - 24\sigma_1(N)\right.\\
&\quad + 24\cdot 1728\sum_{m<\frac{N}{2}}m^2\sigma_5(m)\sigma_1(N-2m) - 24\cdot 1440\cdot T_{m,7,1}(N)\\
&\quad \left. +24\cdot 264\cdot T_{9,1}(N)\right\}q^N.
\end{aligned} \tag{2.11}
$$

And by (1.7) and Proposition 2.1 (c), we obtain

$$
\begin{aligned}
&L(q)L^2(q^2)N(q^2) = L(q)L^2(q^2)\cdot N(q^2)\\
&= \left(1 - 144\sum_{n=1}^{\infty} n^2\sigma_1(n)q^n + 144\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 24\sum_{n=1}^{\infty}\sigma_5(n)q^n\right.\\
&\qquad \left. -2304\sum_{n=1}^{\infty} n^2\sigma_1(n)q^{2n} + 2592\sum_{n=1}^{\infty} n\sigma_3(n)q^{2n} - 480\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\right)\\
&\qquad \times\left(1 - 504\sum_{m=1}^{\infty}\sigma_5(m)q^{2m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\left\{-144N^2\sigma_1(N) + 144N\sigma_3(N) - 24\sigma_5(N) - 576N^2\sigma_1(\frac{N}{2})\right.\\
&\qquad + 1296N\sigma_3(\frac{N}{2}) - 480\sigma_5(\frac{N}{2}) - 504\sigma_5(\frac{N}{2})\\
&\qquad + 144\cdot 504\sum_{m<\frac{N}{2}}(N-2m)^2\,\sigma_1(N-2m)\sigma_5(m)\\
&\qquad - 144\cdot 504\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_3(N-2m)\sigma_5(m)\\
&\qquad + 24\cdot 504\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\sigma_5(m) + 2304\cdot 504\sum_{n<\frac{N}{2}} n^2\sigma_1(n)\sigma_5(\frac{N}{2}-n)\\
&\qquad \left. -2592\cdot 504\sum_{n<\frac{N}{2}} n\sigma_3(n)\sigma_5(\frac{N}{2}-n) + 480\cdot 504\sum_{m<\frac{N}{2}}\sigma_5(\frac{N}{2}-m)\sigma_5(m)\right\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\left\{-144N^2\sigma_1(N) + 144N\sigma_3(N) - 24\sigma_5(N) - 576N^2\sigma_1(\frac{N}{2})\right.\\
&\qquad + 1296N\sigma_3(\frac{N}{2}) - 984\sigma_5(\frac{N}{2}) + 144\cdot 504N^2\cdot T_{5,1}(N)\\
&\qquad - 144\cdot 504\cdot 4N\cdot T_{m,5,1}(N) + 144\cdot 504\cdot 4\sum_{m<\frac{N}{2}} m^2\sigma_5(m)\sigma_1(N-2m)\\
&\qquad - 144\cdot 504N\cdot T_{5,3}(N) + 144\cdot 504\cdot 2\cdot T_{m,5,3}(N) + 24\cdot 504\cdot T_{5,5}(N)\\
&\qquad \left. +2304\cdot 504\cdot I^2_{m,1,5}(\frac{N}{2}) - 2592\cdot 504\cdot I_{m,3,5}(\frac{N}{2}) + 480\cdot 504\cdot I_{5,5}(\frac{N}{2})\right\}q^N.
\end{aligned}
\tag{2.12}
$$

Therefore we equate (2.11) with (2.12) and refer to Proposition 1.1 (d), Proposition 1.2 (c), Proposition 1.3 (b) and (c).

□

3 The induced identities from $\sum_{m<\frac{n}{2}} m^2\sigma_e(m)\sigma_f(n-2m)$

In Theorem 1.2 we obtain some identities induced from Theorem 1.1 :

Proof of Theorem 1.2. (a) Insert Theorem 1.1 (a) into (2.1).

(b) Insert Theorem 1.1 (b) into (2.3).

(c) Insert Theorem 1.1 (c) into (2.5).

(d) Insert Theorem 1.1 (d) into (2.7).

(e) Insert Theorem 1.1 (e) into (2.9).

(f) Insert Theorem 1.1 (f) into (2.11).

□

Now let us consider the convolution sums composing with three divisor functions. In advance, we can see the following proposition in ([7], Theorem 3.11) by the definition of $\sigma_{s,r}(n;m)$ in (1.1) :

Proposition 3.1. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\sum_{m=1}^{n-1} m\sigma_{1,1}(m;2)\sigma_{3,1}(n-m;2)$$
$$= \frac{1}{240}\left[n\left\{\sigma_5(n) + 48\sigma_5(\frac{n}{2}) + 7\sigma_1(n) - 14\sigma_1(\frac{n}{2})\right\} - 8b(n)\right],$$

(b)

$$\sum_{m=1}^{n-1} m\sigma_{1,1}(m;2)\sigma_{3,0}(n-m;2)$$
$$= \frac{1}{90}\left[n\left\{\sigma_5(n) - 22\sigma_5(\frac{n}{2}) - 3\sigma_1(n) + 6\sigma_1(\frac{n}{2})\right\} + 2b(n)\right],$$

(c)

$$\sum_{m=1}^{n-1} m\sigma_{1,0}(m;2)\sigma_{3,0}(n-m;2) = \frac{1}{15}n\left\{7\sigma_5(\frac{n}{2}) - 3n\sigma_3(\frac{n}{2}) - \sigma_1(\frac{n}{2})\right\},$$

(d)

$$\sum_{m=1}^{n-1} m\sigma_{1,0}(m;2)\sigma_{3,1}(n-m;2)$$
$$= \frac{1}{360}\left[n\left\{5\sigma_5(n) - 152\sigma_5(\frac{n}{2}) - 9n\sigma_{3,1}(n;2) + 21\sigma_1(\frac{n}{2})\right\} + 4b(n)\right].$$

Moreover we note that

$$\sum_{m=1}^{n-1} \sigma_1(m)b(n-m) = \frac{1}{48}\left\{32d(n) + c(n) - (3n-2)\,b(n)\right\} \tag{3.1}$$

in ([12], Lemma 3.1 (a)). We construct the convolutions sums by multiplying $\sigma_1(n)$ into the given convolution sums in Proposition 3.1 and obtain Theorem 1.3.

Proof of Theorem 1.3. (a) By Proposition 3.1 (a), we can expand

$$
\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_{1,1}(a;2)\sigma_{3,1}(b;2)\sigma_1(c)\\
&=\sum_{m=1}^{n-1}\sum_{k=1}^{m-1} k\sigma_{1,1}(k;2)\sigma_{3,1}(m-k;2)\sigma_1(n-m)\\
&=\sum_{m=1}^{n-1}\left[\frac{1}{240}\left\{m\left(\sigma_5(m)+48\sigma_5(\frac{m}{2})+7\sigma_1(m)-14\sigma_1(\frac{m}{2})\right)-8b(m)\right\}\right]\sigma_1(n-m)\\
&=\frac{1}{240}\left\{\sum_{m=1}^{n-1} m\sigma_5(m)\sigma_1(n-m)+48\sum_{m<\frac{n}{2}} 2m\sigma_5(m)\sigma_1(n-2m)\right.\\
&\qquad+7\sum_{m=1}^{n-1} m\sigma_1(m)\sigma_1(n-m)-14\sum_{m<\frac{n}{2}} 2m\sigma_1(m)\sigma_1(n-2m)\\
&\qquad\left.-8\sum_{m=1}^{n-1} b(m)\sigma_1(n-m)\right\}\\
&=\frac{1}{240}\left\{I_{m,5,1}(n)+48\cdot 2\cdot T_{m,5,1}(n)+7\cdot I_{m,1,1}(n)-14\cdot 2\cdot T_{m,1,1}(n)\right.\\
&\qquad\left.-8\sum_{m=1}^{n-1} b(m)\sigma_1(n-m)\right\}.
\end{aligned}
$$

So we use Proposition 1.1 (a), (c), (1.16), Proposition 1.3 (b), and (3.1).

(b) By Proposition 3.1 (b), we have

$$
\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_{1,1}(a;2)\sigma_{3,0}(b;2)\sigma_1(c)\\
&=\sum_{m=1}^{n-1}\sum_{k=1}^{m-1} k\sigma_{1,1}(k;2)\sigma_{3,0}(m-k;2)\sigma_1(n-m)\\
&=\sum_{m=1}^{n-1}\left[\frac{1}{90}\left\{m\left(\sigma_5(m)-22\sigma_5(\frac{m}{2})-3\sigma_1(m)+6\sigma_1(\frac{m}{2})\right)+2b(m)\right\}\right]\sigma_1(n-m)\\
&=\frac{1}{90}\left\{\sum_{m=1}^{n-1} m\sigma_5(m)\sigma_1(n-m)-22\sum_{m<\frac{n}{2}} 2m\sigma_5(m)\sigma_1(n-2m)\right.\\
&\qquad-3\sum_{m=1}^{n-1} m\sigma_1(m)\sigma_1(n-m)+6\sum_{m<\frac{n}{2}} 2m\sigma_1(m)\sigma_1(n-2m)\\
&\qquad\left.+2\sum_{m=1}^{n-1} b(m)\sigma_1(n-m)\right\}\\
&=\frac{1}{90}\left\{I_{m,5,1}(n)-22\cdot 2\cdot T_{m,5,1}(n)-3\cdot I_{m,1,1}(n)+6\cdot 2\cdot T_{m,1,1}(n)\right.\\
&\qquad\left.+2\sum_{m=1}^{n-1} b(m)\sigma_1(n-m)\right\}.
\end{aligned}
$$

Thus we use Proposition 1.1 (a), (c), (1.16), Proposition 1.3 (b), and (3.1).

(c) By Proposition 3.1 (c), we obtain

$$\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_{1,0}(a;2)\sigma_{3,0}(b;2)\sigma_1(c)\\
&=\sum_{m=1}^{n-1}\sum_{k=1}^{m-1} k\sigma_{1,0}(k;2)\sigma_{3,0}(m-k;2)\sigma_1(n-m)\\
&=\sum_{m=1}^{n-1}\left\{\frac{1}{15}m\left(7\sigma_5(\frac{m}{2})-3m\sigma_3(\frac{m}{2})-\sigma_1(\frac{m}{2})\right)\right\}\sigma_1(n-m)\\
&=\frac{1}{15}\left\{7\sum_{m<\frac{n}{2}} 2m\sigma_5(m)\sigma_1(n-2m)-3\sum_{m<\frac{n}{2}}(2m)^2\sigma_3(m)\sigma_1(n-2m)\right.\\
&\qquad\left.-\sum_{m<\frac{n}{2}} 2m\sigma_1(m)\sigma_1(n-2m)\right\}\\
&=\frac{1}{15}\left\{7\cdot 2\cdot T_{m,5,1}(n)-3\cdot 4\cdot T^2_{m,3,1}(n)-2\cdot T_{m,1,1}(n)\right\}.
\end{aligned}$$

Thus we refer to (1.16), Proposition 1.3 (b), and Theorem 1.1 (c).

(d) By Proposition 3.1 (d), we have

$$\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_{1,0}(a;2)\sigma_{3,1}(b;2)\sigma_1(c)\\
&=\sum_{m=1}^{n-1}\sum_{k=1}^{m-1} k\sigma_{1,0}(k;2)\sigma_{3,1}(m-k;2)\sigma_1(n-m)\\
&=\sum_{m=1}^{n-1}\frac{1}{360}\left[m\left\{5\sigma_5(m)-152\sigma_5(\frac{m}{2})-9m\sigma_{3,1}(m;2)+21\sigma_1(\frac{m}{2})\right\}+4b(m)\right] \qquad (3.2)\\
&\quad\times\sigma_1(n-m)\\
&=\sum_{m=1}^{n-1}\frac{1}{360}\left[m\left\{5\sigma_5(m)-152\sigma_5(\frac{m}{2})-9m\left(\sigma_3(m)-8\sigma_3(\frac{m}{2})\right)+21\sigma_1(\frac{m}{2})\right\}\right.\\
&\quad\left.+4b(m)\right]\sigma_1(n-m),
\end{aligned}$$

where we used the property of (1.3) and (1.4) for $\sigma_{3,1}(m;2)$. So (3.2) can be written as

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_{1,0}(a;2)\sigma_{3,1}(b;2)\sigma_1(c)$$
$$= \frac{1}{360}\left\{5\sum_{m=1}^{n-1} m\sigma_5(m)\sigma_1(n-m) - 152\sum_{m<\frac{n}{2}} 2m\sigma_5(m)\sigma_1(n-2m)\right.$$
$$-9\sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_1(n-m) + 9\cdot 8\sum_{m<\frac{n}{2}} (2m)^2\sigma_3(m)\sigma_1(n-2m)$$
$$\left. +21\sum_{m<\frac{n}{2}} 2m\sigma_1(m)\sigma_1(n-2m) + 4\sum_{m=1}^{n-1} b(m)\sigma_1(n-m)\right\}$$
$$= \frac{1}{360}\left\{5\cdot I_{m,5,1}(n) - 152\cdot 2\cdot T_{m,5,1}(n) - 9\cdot I^2_{m,3,1}(n) + 9\cdot 8\cdot 4\cdot T^2_{m,3,1}(n)\right.$$
$$\left. +21\cdot 2\cdot T_{m,1,1}(n) + 4\sum_{m=1}^{n-1} b(m)\sigma_1(n-m)\right\}.$$

Thus we use Proposition 1.1 (c), Proposition 1.2 (b), (1.16), Proposition 1.3 (b), Theorem 1.1 (c), and (3.1).

□

4 Conclusions

In this paper we obtain the various convolution sum formulae of

$$T^2_{m,e,f}(n) = \sum_{m<\frac{n}{2}} m^2\sigma_e(m)\sigma_f(n-2m)$$

for $n(\in\mathbb{N})$ and an odd positive integer e and f. Furthermore we have some identities induced from $T^2_{m,e,f}(n)$.

Competing Interests

The author declares that no competing interests exist.

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Appendix

The first twenty values of $\tau(n)$ are given in the Table 1,

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	6	-6048	11	534612	16	987136
2	-24	7	-16744	12	-370944	17	-6905934
3	252	8	84480	13	-577738	18	2727432
4	-1472	9	-113643	14	401856	19	10661420
5	4830	10	-115920	15	1217160	20	-7109760

TABLE 1. $\tau(n)$ for n $(1 \leq n \leq 20)$

similarly the first twenty values of $b(n)$, $c(n)$ and $d(n)$ are listed in the following tables.

n	$b(n)$	n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	6	-96	11	1092	16	4096
2	-8	7	1016	12	768	17	14706
3	12	8	-512	13	1382	18	16344
4	64	9	-2043	14	-8128	19	-39940
5	-210	10	1680	15	-2520	20	-13440

TABLE 2. $b(n)$ for n $(1 \leq n \leq 20)$

n	$c(n)$	n	$c(n)$	n	$c(n)$	n	$c(n)$
1	1	6	2496	11	-38996	16	-65536
2	-16	7	-4536	12	39936	17	311442
3	100	8	-4096	13	37806	18	-74448
4	-256	9	23085	14	15232	19	128244
5	-154	10	-13920	15	-146472	20	-222720

TABLE 3. $c(n)$ for n $(1 \leq n \leq 20)$

n	$d(n)$	n	$d(n)$	n	$d(n)$	n	$d(n)$
1	0	6	-156	11	-536	16	4096
2	1	7	112	12	-2496	17	-17472
3	-8	8	256	13	4384	18	4653
4	16	9	-576	14	-952	19	5848
5	32	10	870	15	336	20	13920

TABLE 4. $d(n)$ for n $(1 \leq n \leq 20)$

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