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$(S,\, S')$-gap Shifts as a Generalization of Run-length-limited Codes

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Abstract

A generalization of Run-length-limited (RLL) codes has been introduced and its dynamical properties as a symbolic dynamical system under the shift map will be investigated. A formula for entropy and zeta function will be given and when our system is shift of finite type, its Bowen-Franks groups are obtained.

Keywords: Shift of finite type; sofic; synchronized; coded system; entropy; zeta function.

2010 Mathematics Subject Classification: 37B10, 37B40, 37A25, 37C30.

1 Introduction

Digital optical discs such as CD, DVD and Blu-ray use extensively Run-length-limited (RLL) codes denoted by $X(d,\, k)$ ([1, 2]). An RLL code limits the run of 0 to be at least d and at most k. A generalization of RLL code is the Maximum Transition Run (MTR) code denoted by MTR$(j,\, k)$ limiting the run of 0 (resp. 1) to be at most k (resp.j) ([3]). A further generalization of these two concepts is the following.

Definition 1.1. Let S and S' be two increasing subsequences of $\mathbb{N}_0$. The coded system Z generated by

$$U = \{0^{s+1}1^{s'+1} : s \in S,\, s' \in S'\}$$

is called the $(S,\, S')$-*gap shift* and is denoted by $X(S,\, S')$.

The system $Z = X(S,\, S')$ is a natural extension of RLL code and a subclass of intertwined systems introduced in [4]. Hence Z is a synchronized system and its simplicity with respect to a general intertwined system will make it possible to determine the Fischer cover for a given S and S' which is done in Section 2. This in turn gives a rather simple routine to compute the entropy and zeta function for Z. When Z is SFT, we also compute the Bowen-Franks groups. These computations are done in Section 3 after discussing some dynamical properties of $(S,\, S')$-gap shifts. The last section is devoted to the conjugacy problem for these systems and we will give some sufficient conditions for two $(S,\, S')$-gap shifts being conjugate or entropy conjugate.

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2 Background and Notations

Let $\mathcal{A}$ be an alphabet, which is a non-empty finite set. In this section, we will bring from [5] some basic definitions of symbolic dynamics on $\mathcal{A}$. The full $\mathcal{A}$-shift denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols in $\mathcal{A}$. A *word (or block)* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. The *empty word*, denoted by ε, includes the sequence of no symbols. If $x \in \mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote a word of length $j - i$ by $x_{[i,\,j]} = x_i x_{i+1} ... x_j$. If $n \geq 1$, then the concatenation of n copies of u denoted by u^n and put $u^0 = \varepsilon$. The shift map σ on $\mathcal{A}^{\mathbb{Z}}$ is defined by $(\sigma(x))_i = x_{i+1}$.

Let $\mathcal{F}$ be a collection of words over $\mathcal{A}$. Define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any word in $\mathcal{F}$. A shift space is a closed subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden words over $\mathcal{A}$. If $\mathcal{F}$ is finite, then $X_{\mathcal{F}}$ is called *shift of finite type* (SFT).

Let $\mathcal{B}_n(X)$ denote the set of all admissible words of length n. The *Language* of X is the collection $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$. A shift space X is *irreducible* if for $u, v \in \mathcal{B}(X)$, there is $w \in \mathcal{B}(X)$ so that $uwv \in \mathcal{B}(X)$. It is *mixing* if for $u, v \in \mathcal{B}(X)$, there is an N such that for each $n \geq N$ there is a word $w \in \mathcal{B}_n(X)$ such that $uwv \in \mathcal{B}(X)$. The word $v \in \mathcal{B}(X)$ is *synchronizing* if whenever $uv,\, vw \in \mathcal{B}(X)$, implies that $uvw \in \mathcal{B}(X)$. An irreducible shift space X is a *synchronized system* if it has a synchronizing word ([6]).

Let $\mathcal{A}$ and $\mathcal{D}$ be alphabets and X a shift space over $\mathcal{A}$. Fix integers m and n with $m \leq n$. Define the $(m + n + 1)$*-block map* $\Phi : \mathcal{B}_{m+n+1}(X) \to \mathcal{D}$ by

$$y_i = \Phi(x_{i-m} x_{i-m+1} ... x_{i+n}), \tag{2.1}$$

where y_i is a symbol in $\mathcal{D}$. The map $\phi = \Phi_{\infty}^{[m,\,n]} : X \to \mathcal{D}^{\mathbb{Z}}$ defined by $y = \phi(x)$ whose ith coordinate given by (2.1) is called the *sliding block code* with *memory* m and *anticipation* n induced by Φ. An onto sliding block code $\phi : X \to Y$ is called a factor code. If the map ϕ is invertible, it is a *conjugacy*.

An *edge shift*, denoted by X_G, is a shift space consisting of all bi-infinite walks in a directed graph G. Let $\mathcal{E} = \mathcal{E}(G)$ be the *edge set* of G and $\mathcal{V} = \mathcal{V}(G)$ the *vertices* or *states* of G. A *labeled graph* $\mathcal{G}$ is a pair $(G, \mathcal{L})$ where $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ is the labeling. Let $\mathcal{L}_{\infty}(X_G) = \{\mathcal{L}_{\infty}(\xi) : \xi \in X_G\}$. Then our subshift $X = X_{\mathcal{G}} = \overline{\mathcal{L}_{\infty}(X_G)}$ and we say $\mathcal{G}$ is a *cover* of X. If G is finite, then X is called *sofic* and any sofic system is synchronized.

A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex $I \in \mathcal{V}(G)$ the edges starting at I carry different labels. It is *right-closing* with delay D if whenever two paths of length $D + 1$ starting at the same vertex and having the same label, have the same initial edge. Similarly, *left-closing* will be defined. A labeled graph is *bi-closing*, if it is simultaneously right-closing and left-closing. An irreducible sofic shift is called *almost-finite-type* (AFT) if it has a bi-closing cover.

Call $F(I) = \{u : u$ is the label of some paths starting at $I\}$ the *follower set* of I. Also the *follower set* of $w \in \mathcal{B}(X)$ is $F(w) = \{v \in \mathcal{B}(X) : wv \in \mathcal{B}(X)\}$. The labeled graph $\mathcal{G}$ is *follower-separated* if distinct vertices have distinct follower sets. A shift space X is sofic if and only if it has a finite number of follower sets. In this case, we have a labeled graph $\mathcal{G} = (G, \mathcal{L})$ called the *follower set graph* of X. The vertices of G are the follower sets and if $wa \in \mathcal{B}(X)$, then draw an edge labeled a from $F(w)$ to $F(wa)$. If $wa \notin \mathcal{B}(X)$, then do nothing.

A *minimal right-resolving cover* of a sofic shift X, also called the *Fischer cover*, is a right-resolving cover of X having the least vertices among all right-resolving covers of X. A right-resolving graph $\mathcal{G}$ is the Fischer cover of X if and only if it is irreducible and follower-separated.

Now let X be a not necessarily sofic system and $x \in \mathcal{B}(X)$. Then $x_+ = (x_i)_{i \in \mathbb{Z}^+}$ (resp. $x_- = (x_i)_{i \in \mathbb{Z}^-}$) is called *right (resp. left) infinite X-ray*. For a left infinite X-ray, say x_-, its follower set is $\omega_+(x_-) = \{x_+ \in X^+ : x_- x_+$ is a point in $X\}$. Consider the collection of all follower sets $\omega_+(x_-)$ as the set of vertices of a graph X^+. There is an edge from I_1 to I_2 labeled a if and only if there is an X-ray x_- such that $x_- a$ is an X-ray and $I_1 = \omega_+(x_-)$, $I_2 = \omega_+(x_- a)$. This labeled graph is called the *Krieger graph* for X. If X is a synchronized system with synchronizing word α, the irreducible component of the Krieger graph containing the vertex $\omega_+(\alpha)$ is called the *right Fischer cover* of X ([7]).

The entropy of a shift space X is defined by $h(X) = \lim_{n\to\infty}(1/n)\log|\mathcal{B}_n(X)|$. Other entropies will be used in this note as well. One is the entropy $h(G)$ for a directed graph G given by Gurevich and defined as

$$h(G) = \lim_{n\to\infty}\frac{1}{n}\log B_{IJ}(n), \tag{2.2}$$

where $B_{IJ}(n)$ is the number of paths of length n starting at vertex I and ending at vertex J ([8, Proposition 1.2]).

3 Fischer Cover of $X(S,\, S')$

In the sequel we will construct the Fischer cover of Z from the Fischer covers of $X(S)$ and $X(S')$, which we will need in this note. It turns out that the most involved case is when $X(S)$ and $X(S')$ are sofic. So we require some more detailed information for a sofic $X(S)$ and thus we borrow some facts and notations for such systems from [9]

If $X(S)$ is sofic, then by [10, Theorem 3.4], one has

$$\Delta(S) = \{d_1,\, d_2, \dots,\, d_k, \overline{g_1,\, g_2, \dots,\, g_l}\}, \qquad g = \sum_{i=1}^{l} g_i, \tag{3.1}$$

where $d_1 = s_1$, $d_i = s_i - s_{i-1}$, $2 \le i \le k$ and $g_j = s_{k+j} - s_{k+j-1}, 1 \le j \le l$. Here k and l are the least integers such that (3.1) holds. Also

$$\mathcal{V}_S = \{F(1),\, F(10), \cdots,\, F(10^{n(S)})\} \tag{3.2}$$

is the set of follower sets which one may consider as the set of vertices for the Fischer cover of $X(S)$ where $n(S)$ is defined in the following.

Definition 3.1.

1. Suppose $X(S)$ is sofic. If $|S| < \infty$, then set $n(S) = |S|$. If $|S| = \infty$, then $n(S)$ will be defined as follows.
   (a) For $k = 1$ and $g_l > s_1$,
      i. if $g_l = s_1 + 1$, then $F(10^{s_l+1}) = F(1)$ and $n(S) = s_l$ (Figure 1);
      ii. if $g_l > s_1 + 1$, then $F(10^{g}) = F(1)$ and $n(S) = g - 1$ (Figure 2).
   (b) For $k \ne 1$, if $g_l > d_k$, then $F(10^{g+s_{k-1}+1}) = F(10^{s_{k-1}+1})$ and $n(S) = g + s_{k-1}$ (Figure 3).
   (c) For $k \in \mathbb{N}$, if $g_l \le d_k$, then $F(10^{s_{k+l-1}+1}) = F(10^{s_k - g_l+1})$ and $n(S) = s_{k+l-1}$ (Figure 4).
2. If $X(S)$ is not sofic, then $n(S) = \infty$.

3.1 Constructing Fischer cover of $X(S,\, S')$ from $X(S)$ and $X(S')$

Suppose

$$S = \{s_i\}_{1\le i\le n(S)} \quad \text{and} \quad S' = \{s'_i\}_{1\le i\le n(S')} \tag{3.3}$$

are two increasing sequences in $\mathbb{N}_0$ and let $\mathcal{G}_S$ and $\mathcal{G}_{S'}$ be the Fischer covers for shifts $X(S)$ and $X(S')$ respectively. Note that the main difference between a S-gap shift and a S'-gap shift is that the former restricts the number of 0's between two 1's whereas the latter restricts the number of 1's between two 0's. The vertices for the Fischer cover of a S-gap shift given in (3.2) where $n(S)$ is defined in Definition 3.1. See Figures 1, 2, 3 and 4. Similar arguments show that such vertices for a S'-gap shift is $\mathcal{V}_{S'} = \{F_{S'}(0),\, F_{S'}(01),\, \dots,\, F_{S'}(01^{n(S')})\}$ where $n(S')$ is defined accordingly. When $X(S)$ is not sofic, no different case arises for Fischer cover: any edge from $F_S(10^i)$ to $F_S(10^{i+1})$ is labeled 0 and edges from $F_S(10^s)$, $s \in S$ to $F_S(1)$ is labeled 1.

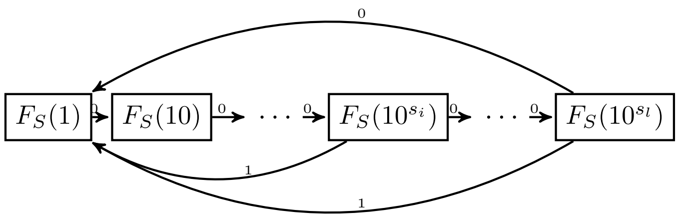

Figure 1: A Fischer cover representing Definition 3.1 (1a(i)).

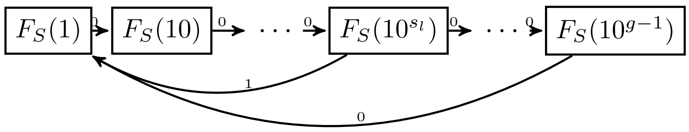

Figure 2: A Fischer cover representing Definition 3.1 1(a)ii.

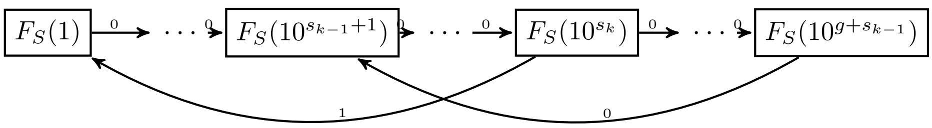

Figure 3: A Fischer cover representing Definition 3.1 (1b).

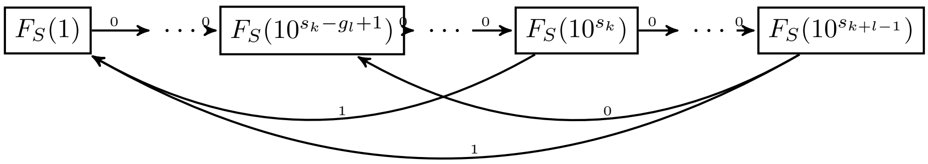

Figure 4: A Fischer cover representing Definition 3.1 (1c).

First arrange the vertices of the Fischer covers of $X(S)$ and $X(S')$ in two rows. Then rename the vertex $F_S(10^i)$ (resp. $F_{S'}(01^j)$) to $F_Z(10^{i+1})$ (resp. $F_Z(01^{j+1})$), $0 \le i \le n(S)$ (resp. $0 \le j \le n(S')$). Recall that the terminating vertex of any edge labeled 1 (resp. 0) is $F_S(1)$ (resp. $F_{S'}(0)$) in $\mathcal{G}_S$ (resp. $\mathcal{G}_{S'}$) now turning to $F_Z(10)$ (resp. $F_Z(01)$). To construct $\mathcal{G}_Z$, cut off these edges from $F_Z(10)$ (resp. $F_Z(01)$) and glue them to $F_Z(01)$ (resp. $F_Z(10)$). Now $\mathcal{G}_Z$ is a cover but not necessarily a Fischer cover. See Figure 5 for the case where $X(S)$ and $X(S')$ are SFT.

Theorem 3.1. *$\mathcal{G}_Z$ is the Fischer cover for $Z = X(S,\, S')$, when*

1. *$X(S)$ and $X(S')$ are sofic and*
   (a) *S and S' are finite.*
   (b) *S is finite and S' does not satisfy 1(a)ii in Definition 3.1.*
   (c) *S and S' are infinite and either S or S' does not satisfy 1(a)i in Definition 3.1.*
   In above cases, $\mathcal{G}_Z$ is finite and if it is not the Fischer cover for $Z = X(S,\, S')$, then the cover $\mathcal{H}$ obtained from $\mathcal{G}_Z$ by merging $F_Z(10^{n(S)+1})$ and $F_Z(01^{n(S')+1})$ is the Fischer cover where $n(S)$ is defined in Definition 3.1.
2. *$X(S)$ or $X(S')$ is non-sofic.*

Proof. The cover $\mathcal{G}_Z = (G_Z,\, \mathcal{L}_Z)$ is an irreducible right-resolving cover of Z but not necessarily minimal. The merged graph of $\mathcal{G}_Z$, say $\mathcal{H}$, is minimal and is the Fischer cover of Z. We show that $\mathcal{H} = \mathcal{G}_Z$ for our cases by showing that $\mathcal{G}_Z$ is follower separated. In fact we exclude cases where $\mathcal{G}_Z$ is not follower-separated.

Vertices in either rows of G_Z correspond to vertices of $\mathcal{G}_S$ and $\mathcal{G}'_S$, so the follower sets representing vertices cannot be equal. Hence we look for those equal vertices in different rows. That is it is sufficient to consider the equality amongst a vertex $v = F(10^{s+1})$, $s \in S$ and $v' = F(01^{s'+1})$, $s' \in S'$. As in (3.1), let

$$\Delta(S') = \{d'_1,\, d'_2, \ldots,\, d'_{k'}, \overline{g'_1,\, g'_2, \ldots,\, g'_{l'}}\} \tag{3.4}$$

and $g' = \sum_{i=1}^{l'} g'_i$.

1. (a) Suppose S and S' are finite and let $S = \{s_1, \ldots,\, s_k\}$ and $S' = \{s'_1, \ldots,\, s'_{k'}\}$. Fix $1 \le i \le k$ and first let $i \ne k$ and set $v = F_Z(10^{s_i+1})$. Then v can be only equal to a $v' = F_Z(01^{s'_j+1})$ for some $s'_j \in S' \setminus \{s'_l\}$. But $0^{s_k+1} \in F_Z(01^{s'+1})$ for all $s' \in S' \setminus \{s'_l\}$ and 0^{s_k+1} is not in v as a follower set and consequently equality does not happen. Therefore, we let $i = k$ and we notice that in this case the out-degree of v is one with label 1. Thus v can only be equal to $v' = F_Z(01^{j_1+1})$ for some $j_1 \notin S'$. However, $1^{s'_l+1} \in F_Z(10^{s_k+1}) \setminus F_Z(01^{j+1})$ for all $j \notin S'$ and this implies $\mathcal{G}_Z$ is follower separated.
   (b) Suppose S is finite and $|S'| = \infty$. We show that unless $n(S')$ satisfies 1(a)ii $\mathcal{G}_Z$ is follower-separated.

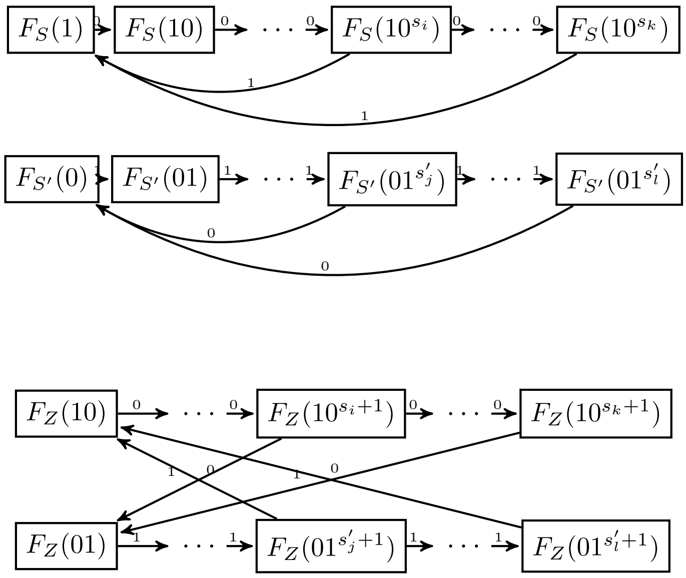

Figure 5: From above, the Fischer covers of $X(S)$, $X(S')$ and $X(S, S')$ respectively.

Suppose $v = v'$. Thus the out-degree and labelings of outer edges of v and v' must be the same and first, suppose their out-degree is one and call the associated edges e_v and $e_{v'}$ respectively. If $v = F_Z(10^i)$ (resp. $v' = F_Z(01^j)$), $1 \leq i < n(S) + 1$ (resp. $1 \leq j \leq n(S') + 1$), then e_v (resp. $e_{v'}$) is labeled 0 (resp. 1) and if $v = F_Z(10^{n(S)+1})$, then e_v is labeled 1. In order to have $v = v'$, one must have v being the far right vertex in its row. The situation where $|S| < \infty$, $|S'| = \infty$ and $X(S')$ satisfies 1(a)ii is the only case with $v = v'$. In this situation, e_v and $e_{v'}$ both terminate at $F_Z(01)$ and are labeled 1 and this implies that both have the same follower sets and merging is required.

If the out-degree is two for v and v', then $v = F_Z(10^{s+1})$ for $s \in S \setminus \{s_k\}$ and $v' = F_Z(01^{s'+1})$ for $s' \in S'$. But then $0^{s_k+1} \in F_Z(01^{s'+1}) \setminus F_Z(10^{s+1})$ and so $v \neq v'$.

(c) Suppose both S and S' are infinite and assume $v = v'$. Recall that when $|S| = \infty$ (resp. $|S'| = \infty$), any edge starting from a vertex with the out-degree one, is labeled 0 (resp. 1). So the out-degrees of v and v' must be two. Let $v = F_Z(10^{s_i+1})$. Then there is a path labeled $0^{s_j - s_i}1$, $i \leq j \leq k + l$ starting at v. This implies there exists a path with the same label starting at v' as well which contradicts the fact that k was the least integer. Thus $\mathcal{H} = \mathcal{G}_Z$ and no merging is needed. It remains to check for $v = F_Z(10^{n(S)+1}) = F_Z(01^{n(S')+1}) = v'$ and both with out-degree 2. In fact, for the case where both $X(S)$ and $X(S')$ satisfy 1(a)i, $v = v'$: both v and v' have one edge with label 0 and terminating at the same vertex $F_Z(10)$ and one edge with label 1 and terminating at the same vertex $F_Z(01)$. Again merging is required.

2. The above proof showed that merging is only required for the two far right vertices in different rows. Such a vertex does not exist if $\mathcal{G}_S$ or $\mathcal{G}_{S'}$ is infinite.

□

3.2 Adjacency Matrix for $X(S, S')$

When S and S' are finite, $Z = X(S, S')$ is SFT and so the adjacency matrix A_Z carries most of the information about the system. Here we use this matrix to give the Bowen-Franks groups of $X(S, S')$. One way to obtain A_Z is to use the Fischer cover given by Theorem 3.1. So A_Z will be the $(n(S)+n(S')+2) \times (n(S)+n(S')+2)$ matrix

$$A_Z = \begin{pmatrix} A^1 & 0 \\ A^2 & A^3 \end{pmatrix} \tag{3.5}$$

where A^1 and A^3 are $(n(S)+1) \times (n(S)+2)$, $(n(S')+1) \times n(S')$ matrices respectively and nonzero entries are as follows.

1. $A^1_{i(i+1)} = A^1_{j(n(S)+2)} = 1$ for $1 \leq i \leq n(S)+1$, $j \in S \setminus \{n(S)+1\}$.
2. $A^2_{(i+1)1} = 1$ for $i \in S'$.
3. $A^3_{ii} = 1$ for $1 \leq i \leq n(S')$.

4 Dynamical Properties of (S, S')-gap Shifts

In this section we investigate the dynamical properties of (S, S')-gap shifts in terms of S and S'. We also give a formula for computing the entropy and zeta function of these systems. So far we know that (S, S')-gap shifts are all synchronized and so coded generated by $\{0^{s+1}1^{s'+1} : s \in S, s' \in S'\}$.

Theorem 4.1. *The following are equivalent for a (S, S')-gap shift Z.*

1. *Z is mixing;*
2. *$\gcd\{s_n + s'_m + 2 : s_n \in S, s'_m \in S'\} = 1$.*

Proof. $1 \Rightarrow 2$. If Z is mixing, then there exists $N > 0$ such that for all $n \geq N$, there exists $w \in \mathcal{B}_n(Z)$ with $01w01 \in \mathcal{B}(Z)$. Thus there are words of length $N+1$ and $N+2$ of the form $1^{s'_{i_1}+1}0^{s_{j_1}+1}1^{s'_{i_2}+1}0^{s_{j_2}+1}\cdots 1^{s'_{i_r}+1}0^{s_{j_r}+1}$ with $s_j \in S, s'_i \in S'$ implying that $\gcd\{s_n + s'_m + 2 : s_n \in S, s'_m \in S'\} = 1$.

$2 \Rightarrow 1$. If $\gcd\{s_n + s'_m + 2 : s_n \in S, s'_m \in S'\} = 1$, then for all sufficiently large n, there is a word of length n of the form $01^{s'_{i_1}+1}0^{s_{j_1}+1}1^{s'_{i_2}+1}\cdots 0^{s_{j_r}+1}1$. Since 01 is synchronizing and Z is irreducible, the result follows. □

Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph, I a vertex of G and $A = A_G$ the associated adjacency matrix. The follower set $F_{\mathcal{G}}(I)$ of I in G is the collection of all labels of paths starting at I. The *period of a vertex* I, denoted by $\mathrm{per}(I)$, is the greatest common divisor of those integers $n \geq 1$ for which $(A^n)_{II} > 0$. The *period of the matrix* A denoted by per(A) is the greatest common divisor of all the numbers per(J) where J is a vertex. If A is irreducible, then all vertices have the same period. The *period of a graph* G is the period of its adjacency matrix and is denoted by per(G). Let X_G be an irreducible edge shift and $p = \mathrm{per}(G)$. Then there exists a unique partition $\{D_0, D_1, ..., D_{p-1}\}$ of the vertices of G, called *period classes*, so that every edge starting in D_i terminates in $D_{(i+1) \mod p}$. Recall that for a S-gap shift, per$(G) = \gcd(S+1)$ where G is the underlying graph of Fischer cover of $X(S)$.

Lemma 4.2. *Suppose $X(S, S')$ is a sofic shift with the Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Then* per$(G) = \gcd(S + S' + 2)$.

Proof. Let A be the adjacency matrix of G and $I = F(10)$. Starting at I on G, we are back again at I if we have traced on G a path labeled $0^{s_i}1$, $s_i \in S$ followed by a path labeled $1^{s'_j}0$, $s'_j \in S'$. Therefore,

$$\{n : (A^n)_{II} > 0, n \in \mathbb{N}\} = \left\{\sum(s_i + s'_j + 2) : s_i \in S, s'_j \in S'\right\}. \quad (4.1)$$

On the other hand,

$$\gcd\left\{\sum(s_i + s'_j + 2) : s_i \in S, s'_j \in S'\right\} = \gcd\left\{s + s' + 2 : s \in S, s' \in S'\right\}. \quad (4.2)$$

So (4.1) and (4.2) imply that per$(I) = \gcd(S + S' + 2)$. Since A is irreducible, we are done. □

The rest of this section is devoted to other similarities of (S, S')-gap shifts with S-gap shifts. We srart with the following theorem which characterizes some properties of $X(S, S')$ in terms of the combinatorial properties of S and S'; compare [11, Example 3.4] and [10, Theorems 3.3, 3.4 and 3.6]. Before, recall that $\Delta(S) = \{d_n\}_n$ and $\Delta(S') = \{d'_n\}_n$ where $d_1 = s_1$, $d'_1 = s'_1$, $d_n = s_n - s_{n-1}$ and $d'_n = s'_n - s'_{n-1}$, $n \geq 2$. Note that $\Delta(S)$ (resp. $\Delta(S')$) here is in consistence with (3.1) (resp. (3.4)).

Theorem 4.3. *$X(S, S')$ is*

1. *SFT if and only if S and S' are finite or cofinite.*
2. *sofic if and only if $\Delta(S)$ and $\Delta(S')$ are eventually periodic.*
3. *AFT if and only if $\Delta(S)$ and $\Delta(S')$ are eventually constant.*
4. *SVGL if and only if $\sup_i |s_{i+1} - s_i| < \infty$ and $\sup_i |s'_{i+1} - s'_i| < \infty$.*

Proof. The necessity and sufficient conditions for a S-gap shift being SFT, sofic and AFT was given in [10]. Also, a necessity and sufficient condition for a SVGL S-gap is in [11, Example 3.5]. Using those results, (1), (2), (4) and the necessity condition for (3) will be deduced from respective results in [4, Theorems 3.3, 3.5 and 3.6]. So it remains to prove the sufficiency of part (3). Suppose $\Delta(S)$ and $\Delta(S')$ are eventually constant. Then

$$\Delta(S) = \{d_1, d_2, ..., d_k, \overline{g}\}, \qquad \Delta(S') = \{d'_1, d'_2, ..., d'_{k'}, \overline{g'}\} \quad (4.3)$$

where $g = s_{k+1} - s_k$ and $g' = s'_{k'+1} - s'_{k'}$. A proof is established by showing that the cover of $X(S, S')$ given in subsection 3.1 is left-closing by giving a delay.

If it is not left-closing, then there are two paths ξ_- and ξ'_- labeled the same and terminating at the same vertex. Noticing the cover of our $X(S, S')$ given in subsection 3.1, the vertices with more than one inner edges are $F(10)$, $F(01)$, $F(10^{(n(S)+2) \mod g})$ and $F(01^{(n(S')+2) \mod g'})$.

The label of a path of length $s'_{k'} + 2$ (resp. $s_k + 2$) terminating at $F(10)$ (resp. $F(01)$) determines its ending edge. On the other hand, The label of a path of length $(n(S) + 2)(\mod g) + 1$ (resp. $(n(S')+2)(\mod g')+1$) terminating at $F(10^{(n(S)+2) \mod g})$ (resp. $F(01^{(n(S')+2) \mod g'})$) determines its ending edge. So the cover has $D = \max\{s_k + 2, s'_{k'} + 2, (n(S) + 2)(\mod g) + 1, (n(S') + 2)(\mod g') + 1\}$ as its delay and the proof is complete. □

An irreducible sofic shift space X is *near Markov* when it is AFT and its derived shift space ∂X is finite ([12]).

Theorem 4.4. *Any AFT (S, S')-gap shift is near Markov.*

Proof. Let $Z = X(S, S')$ be a strictly AFT shift. Two words 10 and 01 are synchronizing words. Hence, points having both 0 and 1 as some of its entries are not in ∂Z and $\partial Z \subseteq \{0^\infty, 1^\infty\}$. So ∂Z is finite. □

Set $\mathcal{F}$ to be a finite collection of words over a finite alphabet $\mathcal{A}$ where each $w_j \in \mathcal{F}$ is associated with a non-negative integer index n_j. Write

$$\mathcal{F} = \{w_1^{(n_1)}, w_2^{(n_2)}, ..., w_{|\mathcal{F}|}^{(n_{|\mathcal{F}|})}\} \tag{4.4}$$

and associate with the indexed list $\mathcal{F}$ a period T, where T is a positive integer satisfying $T \geq \max\{n_1, n_2, ..., n_{|\mathcal{F}|}\} + 1$.

A shift space X is a shift of *periodic-finite-type* (PFT) if there exists a pair $\{\mathcal{F}, T\}$ with $|\mathcal{F}|$ and T finite so that $X = X_{\{\mathcal{F},T\}}$ is the set of bi-infinite sequences that can be shifted such that the shifted sequence does not contain a word $w_j^{n_j} \in \mathcal{F}$ starting at any index m with $m \mod T = n_j$. A strictly PFT shift cannot be represented as an SFT.

Let $\mathcal{G}$ be the Fischer cover of an irreducible sofic shift, $p = \mathrm{per}(A_{\mathcal{G}})$ and $D_0, D_1, ..., D_{p-1}$ the period classes of $\mathcal{G}$. An indexed word $w^{(n)} = (w_0, w_1, ..., w_{l-1})^{(n)}$ is a *periodic first offender of period class* n if $w \notin \cup_{I \in D_n} F_{\mathcal{G}}(I)$ but for all $i, j \in [0, l-1]$ with $i \leq j$ and $w_{[i,j]} \neq w$, $w_{[i,j]} \in \cup_{I \in D_{(n+i) \mod p}} F_{\mathcal{G}}(I)$. An irreducible sofic shift is PFT if and only if the set of periodic first offenders is finite ([13, Corollary 14]).

A S-gap shift is strictly PFT if and only if it is AFT and non-mixing ([10, Theorem 3.7]). A main ingredient of the proof for this fact given in [10, Theorem 3.7] is defining a set of periodic first offenders $\mathcal{O} = \{1^{(i)} : 0 \leq i \leq p-2\}$. If one defines $\mathcal{O} = \{(10)^i, (01)^i : 1 \leq i \leq p-1\}$ for (S, S')-gaps, then the same proof will give the following result.

Theorem 4.5. *Suppose $Z = X(S, S')$ is not SFT. Then it is strictly PFT if and only if it is AFT and non-mixing.*

There are also other similarities between S-gap and (S, S')-gap shifts in Theorems 4.8 and 4.9.

Remark 4.1. Not all the properties of $X(S)$ and $X(S')$ transfer to $Z = X(S, S')$ or vice versa. Mixing and PFT are two such properties. As an example, let $S = S' = 2\mathbb{N}$ and $\mathcal{G}_X = (G_X, \mathcal{L}_X)$ be the Fischer cover of X. Then both $X(S)$ and $X(S')$ are mixing ([14]) while $\gcd(S + S') = 2$ and so Z does not have mixing property by Theorem 4.1. Also the same S and S' as above will imply that Z is PFT. This is a consequence of the fact that per$(G_Z) = 2$ (Lemma 4.2)and the fact that the irreducible components of $\mathcal{G}_Z$ and $\mathcal{G}_Z^2$ are definite graphs ([13, Proposition 8]). On the other hand, since per$(G_{X(S)})$ =per$(G_{(X(S')}) = 1$, $X(S)$ and $X(S')$ are not PFT ([15, Proposition 1]).

Now set $S = 3\mathbb{N} - 1$ and $S' = 5\mathbb{N} - 1$. Then

$$\gcd\{s + 1 : s \in S\} = 3, \qquad \gcd\{s' + 1 : s' \in S'\} = 5$$

which means $X(S)$ and $X(S')$ are not mixing ([14]) and since $\Delta(S) = \{2, \overline{3}\}$ and $\Delta(S') = \{4, \overline{5}\}$, both $X(S)$ and $X(S')$ are PFT ([10, Theorem 3.8]). However, $\gcd(S + S' + 2) = 1$ and so Z is mixing (Theorem 4.1) but not PFT ([15, Proposition 1]).

4.1 Entropy and Zeta function of (S, S')-gap Shifts

Let S and S' be the subsets of $\mathbb{N}_0$. If the multiplicity in $S + S'$ is important we will show it by $\{\!\{S + S'\}\!\}$; for instance, if $S = \{1, 3\}$ and $S' = \{2, 4\}$, then $S + S' = \{3, 5, 7\}$ but $\{\!\{S + S'\}\!\} = \{\!\{3, 5, 5, 7\}\!\}$. When no multiplicities exists, we have $S + S' = \{\!\{S + S'\}\!\}$. When S and S' are finite, note that $\{\!\{S + S'\}\!\} = |S||S'|$. We will see that $\{\!\{S + S'\}\!\}$ is a conjugacy invariant and in particular characterizes the entropy.

For a dynamical system (X, T), let p_n be the number of periodic points in X having period n. When $p_n < \infty$, the zeta function $\zeta_T(t)$ is defined as

$$\zeta_T(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n} t^n\right). \tag{4.5}$$

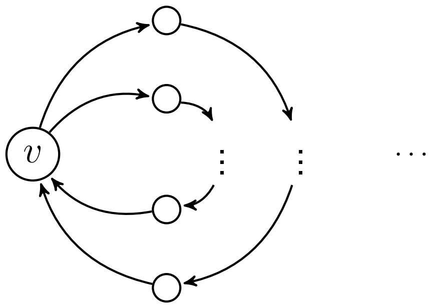

Figure 6: An infinite loop graph.

Then by Taylor's formula,

$$\frac{d^n}{dt^n} \log \zeta_T(t)|_{t=0} = n! \frac{p_n}{n} = (n-1)! p_n. \tag{4.6}$$

A *loop graph* G_ℓ consists of disjoint loops of lengths $l_1, l_2, \ldots$ with $l_i \leq l_{i+1}$ (Figure 6). Let f_n be the number of first-return loops of length n from v to v and define $f = \sum_{n=1}^{\infty} f_n x^n$. Boyle in [16] shows that the zeta function of X_G is

$$\zeta_{\sigma_{G_\ell}}(x) = \frac{1}{1 - f(x)}. \tag{4.7}$$

Theorem 4.6. *Let*

$$g(x) = \sum_{n \in S+S'} \left(\sum_{k+l=n} \chi_S(k) \chi_{S'}(l) \right) x^{(n+2)}. \tag{4.8}$$

1. *The zeta function of a* (S, S')*-gap shift is*

$$\zeta_\sigma(x) = \frac{1}{(1-x)^{p_1(X(S,S'))}(1 - g(x))}.$$

2. *The entropy of a* (S, S')*-gap shift is* $\log \lambda$ *where* λ *is the unique non-negative solution of*

$$g(\frac{1}{x}) = \sum_{s+s' \in \{\!\{S+S'\}\!\}} x^{-(s+s'+2)} = 1. \tag{4.9}$$

Proof. (1) Construct a loop graph G_ℓ with a base vertex at v such that for any $s + s' \in \{\!\{S + S'\}\!\}$, there is a cycle of length $s + s' + 2$ from v to v. The number of cycles of length $n = s + s' + 2$ is $f_n = \sum_{k+l=(n-2) \in S+S'} \chi_S(k) \chi_{S'}(l)$. Therefore, $g(x) = f(x) = \sum_{n=1}^{\infty} f_n x^n$ and by (4.7),

$$\zeta_{\sigma_{G_\ell}}(x) = \frac{1}{1 - g(x)}.$$

Note that all the periodic points in $X(S, S')$ can be represented by a cycle or concatenating some cycles in G_ℓ except possibly 0^∞ and 1^∞, the fix points of $X(S, S')$. Let $0 \leq p_1 \leq 2$ be the number of fix points of $X(S, S')$. If $p_1 = 0$, $\zeta_{\sigma_\mathcal{G}}(x) = \zeta_{\sigma_{G_\ell}}(x)$. If $p_1 = 1$, $p_n(\sigma_\mathcal{G}) = p_n(\sigma_{G_\ell}) + 1$ for all $n \in \mathbb{N}$. So

$$\zeta_{\sigma_\mathcal{G}}(x) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(\sigma_{G_\ell}) + 1}{n} x^n \right) = \frac{1}{(1-x)} \zeta_{\sigma_{G_\ell}}(t).$$

By the same argument, if $p_1 = 2$,

$$\zeta_{\sigma_{\mathcal{G}}}(x) = \frac{1}{(1-x)^2}\zeta_{\sigma_{G_\ell}}(t).$$

(2) Since $Z = X(S,\, S')$ is synchronized, $h(Z) = \max\{h(G), h(\partial Z)\}$ where $h(G)$ is the Gurevich entropy of the Fischer cover of Z ([17, Theorem 6.16]). We prove the theorem by showing that $h(G) = h(G_\ell)$. Let $u = F(10)$ in G. Any cycle starting from u again must travel a path of length $s \in S$ consisting of 0's followed by a path labeled $1^{s'+1}0$, $s' \in S'$. So corresponding to any cycle of length $s + s' + 2$ in G_ℓ, there is a path π of the same length with $i(\pi) = t(\pi) = u$ and vice versa. By (2.2), $h(G) = h(G_\ell)$. On the other hand, $\partial Z \subseteq \{0^\infty, 1^\infty\}$. So $h(Z) = h(G)$. Now a minor alteration of arguments in [11, p. 11] yields the result. □

Recall that two SFT subshifts X and Y with the same zeta function, have the same entropy ([5, Corollary 7.4.13]). This is not the case in a general coded system; however, $(S,\, S')$-gap shifts have this property.

Theorem 4.7. *If $\zeta_{\sigma(S,\,S')}(X(S,\,S')) = \zeta_{\sigma(T,\,T')}(X(T,\,T'))$, then $\{\!\{S + S'\}\!\} = \{\!\{T + T'\}\!\}$. In particular, $h(X(S,\,S')) = h(X(T,\,T'))$.*

Proof. Let $\{\!\{S+S'\}\!\} = \{\!\{v_1, v_2, \cdots\}\!\}$ with $v_i \leq v_{i+1}$, $i \geq 1$. For $n = 1$, $v_1 = s_1+s'_1 = t_1+t'_1 \in \{\!\{T+T'\}\!\}$. Assume for any $n < N$, $v_n = t_m + t'_{m'} \in \{\!\{T + T'\}\!\}$ for some $(t_m,\ t'_{m'}) \in T \times T'$ and consider $v_N = s_{n_0} + s'_{n'_0}$, $(s_{n_0},\, s'_{n'_0}) \in S \times S'$.

By equality of zeta functions and (4.6), we have that for all i, $p_i(X(S,\,S')) = p_i(X(T,\,T'))$; in particular,

$$p_{v_N}(X(S,\,S')) = p_{v_N}(X(T,\,T')). \tag{4.10}$$

Now if $(s_{n_0} + s'_{n'_0}) \notin \{\!\{T + T'\}\!\}$, then (4.10) implies that $s_{n_0} + s'_{n'_0} = \sum_{r=1}^{p}(t_{u_r} + t'_{u'_r})$ with $p > 1$ and $(t_{u_r} + t'_{u'_r}) < v_N$. But any periodic point of period n of $X(S,\,S')$ looks like

$$(1^{s'_{i_1}+1}0^{s_{j_1}+1}1^{s'_{i_2}+1}0^{s_{j_2}+1}\cdots 1^{s'_{i_q}+1}0^{s_{j_q}+1})^\infty \tag{4.11}$$

where $\sum_{r=1}^{q}(s'_{i_r}+s_{j_r}+2) = n$. Thus by our induction assumption, there is a one to one correspondence between those elements of $\{\!\{S + S'\}\!\}$ and $\{\!\{T + T'\}\!\}$ whose values are less than v_N. As a result, we must have $p_{v_N}(X(S,\,S')) \geq p_{v_N}(X(T,\,T')) + 1$ which contradicts (4.10). □

A shift space X is called *almost sofic* if $h(X) = \sup\{h(Y) : Y \subseteq X$ is a sofic subshift$\}$ ([18, Definition 6.7]).

Theorem 4.8. *Every $(S,\,S')$-gap shift is almost sofic.*

Proof. If $X(S,\,S')$ is sofic, the statement is obvious. Thus suppose $X(S,\,S')$ is not sofic and for every $k \geq 1$, define $S_k = \{s_1, s_2, ..., s_k\}$ and $S'_k = \{s'_1, s'_2, ..., s'_k\}$. Then for all k, $X(S_k,\,S'_k)$ is a sofic subsystem of $X(S,\,S')$ and $\{h(X(S_k,\,S'_k))\}_{k\geq1}$ is an increasing sequence. By (4.9), $h(X(S_k,\,S'_k)) \nearrow h(X(S,\,S'))$ which implies $X(S,\,S')$ is almost sofic. □

A $(S,\,S')$-gap shift is synchronized; so their non-trivial subshift factors are coded with positive entropy ([6]). Moreover, the next theorem shows that every subshift factor of a $(S,\,S')$-gap shift is *intrinsically ergodic*: there is a unique measure of maximal entropy. This fact was established for S-gap shifts by Climenhaga and Thompson in [19].

Theorem 4.9. *1. Every subshift factor of a $(S,\,S')$-gap shift is intrinsically ergodic.*

2. If at least one of the S or S' is infinite and Y is a non-trivial subshift factor of $X(S,\,S')$, then Y has positive entropy.

Proof. (1). Let

$$\mathcal{L}_1 = \{0^k 1^{m_1} 0^{n_1} \cdots 1^{m_p} 0^{n_p} 1^l : m_i - 1 \in S'; n_i - 1 \in S, k, l \in \mathbb{N}\},$$

$$\mathcal{L}_2 = \{1^{k'} 0^{n'_1} 1^{m'_1} 0^{n'_1} \cdots 0^{n'_q} 1^{m'_q} 0^{l'} : m'_j - 1 \in S'; n'_j - 1 \in S, k, k', l' \in \mathbb{N}\}.$$

Then language of $(S,\, S')$-gap shift is $\mathcal{L}_1 \cup \mathcal{L}_2$. So a *uniform CGC-decomposition* for $X(S, S')$ is given by

$$\begin{aligned} \mathcal{G} &= \{0^n 1^m : n-1 \in S, m-1 \in S'\}, \\ \mathcal{C}^p &= \{0^k, 1^k : k \geq 0\}, \\ \mathcal{C}^s &= \{0^l, 1^l : l \geq 1\}. \end{aligned}$$

Let $(\mathcal{C}^p \cup \mathcal{C}^s)_n$ be the words in $\mathcal{C}^p \cup \mathcal{C}^s$ of length n. Then $|(\mathcal{C}^p \cup \mathcal{C}^s)_n| = 2$ for all $n \geq 1$ and so $h(\mathcal{C}_p \cup \mathcal{C}_s) = 0$. Therefore, it follows that every subshift factor of a $(S,\, S')$-gap shift is intrinsically ergodic ([19]).

(2). In [19], Climenhaga and Thompson proved that any shift with uniform CGC-decomposition, either has positive entropy or comprises a single periodic orbit and they also proved that a factor shift of a shift with uniform CGC-decomposition has uniform CGC-decomposition. Now suppose Y is a shift space over the alphabet $\mathcal{A}$ and let $\phi : X(S,\, S') \to Y$ be a factor code with memory m and anticipation n induced by the block map Φ and suppose $|S| = \infty$. Then $\Phi(0^{m+n+1})$ is a symbol in $\mathcal{A}$, say a. Since $|S| = \infty$, the language of Y must contain a^k for any $k \in \mathbb{N}$. This means $Y = \{a^\infty\}$. ☐

4.2 The Bowen-Franks Groups of $(S,\, S')$-gaps

Let A be an $n \times n$ integer matrix. The *Bowen-Franks group* of A is

$$BF(A) = \mathbb{Z}^n / \mathbb{Z}^n(\mathrm{Id} - A),$$

where $\mathbb{Z}^n(\mathrm{Id} - A)$ is the image of $\mathbb{Z}^n$ under the matrix $\mathrm{Id} - A$ acting on the right. See [9, §4], for computing Bowen-Franks group using the *Smith form*.

Note that $BF(A)$ (or denoted by $BF_0(A)$ in some papers) is the cokernel of $\mathrm{Id} - A$ acting on the row space $\mathbb{Z}^n$. The kernel is another Bowen-Franks group $BF_1(A) := \mathrm{Ker}(\mathrm{Id} - A)$. Similarly, acting on the column space $(\mathbb{Z}^n)^t$, $\mathrm{Id} - A$ defines another two groups as cokernel and kernel, denoted by $BF^t(A)$ and $BF_1^t(A)$ respectively. These four groups are called the BF-groups ([20]).

Theorem 4.10. *Let $X(S,\, S')$ be a SFT shift.*

1. *If $|S|, |S'| < \infty$, then $BF(A) \simeq \mathbb{Z}_{(|S||S'|-1)} \simeq BF^t(A)$.*
2. *If $|S| < \infty$ and $|S'| = \infty$, then $BF(A) \simeq \mathbb{Z}_{|S|} \simeq BF^t(A)$.*
3. *If $|S|, |S'| = \infty$, then $BF(A) \simeq \mathbb{Z}_1 \simeq BF^t(A)$.*
4. *$BF_1(A) = BF_1^t(A) = \{0\}$.*

Proof. The adjacency matrix of the Fischer cover of $X(S,\, S')$ is obtained in Subsection 3.2. Then $BF_1(A) = BF_1^t(A) = \{0\}$ is a consequence of the definition. Groups $BF(A)$ and $BF^t(A)$ can be found by computing the Smith forms of $\mathrm{Id} - A$ and $\mathrm{Id} - A^t$ respectively. ☐

Two subshifts are flow equivalent if they have topologically equivalent suspension flows ([5]). Franks in ([21]) classified irreducible SFT's up to flow equivalent by showing that two non-trivial irreducible SFT's X_A and X_B are flow equivalent if and only if $BF(A) \simeq BF(B)$ and $\mathrm{sgn}(\det(\mathrm{Id} - A)) = \mathrm{sgn}(\det(\mathrm{Id} - B))$.

Corollary 4.11. *Suppose $X(S,\, S')$ is a SFT shift.*

1. *If $|S|, |S'| < \infty$, then $X(S,\, S')$ is flow equivalent to full $|S||S'|$-shift.*
2. *If $|S| < \infty$ and $|S'| = \infty$, then $X(S,\, S')$ is flow equivalent to full $(|S| + 1)$-shift.*
3. *If $|S|, |S'| = \infty$, then $X(S,\, S')$ is flow equivalent to full 2-shift.*

The Bowen-Franks groups for an RLL shift was announced in [9]. By above corollary, for an asymmetric-RLL $X = X(d_1,\, k_1,\, d_0,\, k_0)$ constraint with the adjacency matrix A,

$$BF(A) \simeq \mathbb{Z}_{(k_0-d_0+1)(k_1-d_1+1)-1} \simeq BF^t(A)$$

and X is flow equivalent to full $(k_0 - d_0 + 1)(k_1 - d_1 + 1)$-shift.

5 The Conjugacy Problem for $(S,\, S')$-gap Shifts

The conjugacy problem has been solved completely for S-gap shifts ([10]). Here, we give a general necessary condition and some sufficient conditions for special cases.

The following result is a consequence of Theorem 4.7.

Theorem 5.1. *If $X(S,\, S')$ and $X(T,\, T')$ are conjugate, then $\{\!\{S + S'\}\!\} = \{\!\{T + T'\}\!\}$.*

The converse is not true. For instance, let $S = \{1, 2\}$, $S' = 2\mathbb{N}_0+1$, $T = \{1\}$ and $T' = \{1, 2, \cdots\}$. Then $\{\!\{S + S'\}\!\} = \{\!\{T + T'\}\!\}$. By Theorem 4.6, $h(X(S,\, S')) = h(X(T,\, T'))$. But $X(S,\, S')$ is a strictly sofic shift while $X(T,\, T')$ is a SFT shift, so they are not conjugate. However, in sequel we show that $\{\!\{S + S'\}\!\} = \{\!\{T + T'\}\!\}$ implies entropy-conjugacy.

Also,

$$\zeta_{X(S,\,S')}(t) = \zeta_{X(T,\,T')}(t) = \frac{1}{1 - t - t^4}.$$

Thus unlike S-gap shifts ([10, Corollary 4.2]), the zeta function is not a complete invariant for conjugacy.

Theorem 5.2. *If $\{\!\{S + S'\}\!\} = \{\!\{T + T'\}\!\}$, then $X(S,\, S')$ and $X(T,\, T')$ are conjugate in the following cases.*

1. *$s_i + s'_j = t_i + t'_j$ for all i and j,*
2. *$X(S,\, S')$ and $X(T,\, T')$ are SFT and $S + S' = \{\!\{S + S'\}\!\} = \{\!\{T + T'\}\!\} = T + T'$. In particular, they are conjugate to a S-gap shift $X(S'')$, $S'' = S + S' + 1$.*

Proof. (1). Without loss of generality assume $s'_1 > t'_1$. Let $n = s'_1 - t'_1 + 1$ and define $\Phi : \mathcal{B}_n(X(S,\, S')) \to \{0, 1\}$,

$$\Phi(w) = \begin{cases} 1 & w = 1^n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi = \Phi_\infty^{[0,\,n-1]} : X(S,\, S') \to X(T,\, T')$ is a conjugacy map. In fact, what ϕ does is to replace $1^{s'_i+1}0^{s_j+1}1^n$ with $1^{t'_i+1}0^{t_j+1}$. This is because the word $1^{s'_i+1}$ contains $(s'_i + 1 - (n - 1))$ times the word 1^n; the first starting from the first position of $1^{s'_i+1}$, the second from the second position and so forth. But $s'_i + 1 - (n - 1) = s'_i - s'_1 + t'_1 + 1$ and also by (1),

$$s_1 + s'_i = t_1 + t'_i, \qquad s_1 + s'_1 = t_1 + t'_1.$$

So $s'_i - s'_1 + t'_1 = t'_i$. By the same reasoning, $(n - 1) + s_j + 1 = t_j + 1$ and so we are done.

(2). A conjugacy map can be set via a S-gap shift $X(S'')$ as follows. First define $\Phi : \mathcal{B}_2(X(S,\, S')) \to \{0, 1\}$,

$$\Phi(w) = \begin{cases} 1 & w = 10, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Now let $S'' = S + S' + 1$ and $\phi : X(S, S') \to X(S'')$ be the sliding block code with memory 0 and anticipation 1 induced by Φ. Then ϕ defines a conjugacy map. Also, $X(T, T')$ and $X(S'')$ are conjugate with the same conjugacy map by letting $\Phi : \mathcal{B}_2(X(T, T')) \to \{0, 1\}$ be defined as (5.1). $\square$

Which (S, S')-gap shifts, as a generalization for the Run-length-limited (RLL) shifts, are conjugate to a given RLL shift?

Although the problem of conjugacy for (S, S')-gap shifts has not been sorted out for general settings, in most cases there are several (S, S')-gap shifts conjugating to a given RLL shift. For instance, let $|S| = p$ and $|S'| = p'$. If $X(S, S')$ and an RLL shift $X(d, k)$ are conjugate, then by Theorem 5.1, $\{\!\{S + S'\}\!\} = \{d-1, d-2, \ldots, k-1\}$. So we have

$$d = s_1 + s'_1 + 1, \quad \text{and} \quad k = s_p + s'_{p'} + 1 \tag{5.2}$$

where s_1, s'_1 are minimum and s_p, $s'_{p'}$ are maximum of the respective spaces.

Now suppose d and k are the natural numbers such that $k > d$ and $k - d + 1$ is not prime. Also set $k - d + 1 = p \times p'$, $1 < p \leq p'$. Then RLL shift $X(d, k)$ and $X(S, S')$ are conjugate for some

$$S = \{s_1, s_2, \cdots, s_p\}, \quad \text{and} \quad S' = \{s'_1, s'_2, \cdots, s'_{p'}\} \tag{5.3}$$

satisfying (5.2) and

$$s_{i+1} - s_i = p', \qquad s'_{j+1} - s'_j = 1$$

for $1 \leq i \leq p-1$ and $1 \leq j \leq p'-1$. The conjugacy follows from the case (2) of Theorem 5.2. So other (S, S')-gap shifts conjugate to $X(d, k)$ may be found when $\{\!\{S+S'\}\!\}$ equals $\{d-1, d-2, \ldots, k-1\}$.

Corollary 5.3. *1. If $X(S, S')$ and an RLL shift $X(d, k)$ are conjugate, then S and S' are SFT with $|S| = p$ and $|S'| = p'$ for some $p, p' \in \mathbb{N}$ and d and k satisfies* (5.2)*.*

2. *Suppose $X(d, k)$ is an RLL shift with $k - d + 1 = p \times p'$, $1 < p \leq p'$. Then $X(d, k)$ and $X(S, S')$ are conjugate for some S and S' as in* (5.3)*.*
3. *Let $p \in \mathbb{N}$ be a prime number and $X(d, k)$ be an RLL shift with $k - d + 1 = p$. Then $X(d, k)$ is conjugate to a $X(S, S')$, with $S = \{s\}$, $s \leq d-1$ and $S' = \{d-s-1, d-s, \ldots, k-s-1\}$.*

Example 5.4. *The RLL shift $X(2, 9)$ and $X(S, S')$ are conjugate for*

1. $S = \{1, 5\}$, $S' = \{1, 2, 3, 4\}$,
2. $S = \{1, 2\}$, $S' = \{1, 3, 5, 7\}$,
3. $S = \{1, 3\}$, $S' = \{1, 2, 5, 6\}$.

The first two are implied by Corollary 5.3(2) and the third where S and S' do not satisfy the conditions of S and S' in 5.3 is a consequence of Theorem 5.2(2).

A set $E \subseteq X$ is *entropy-negligible* for X if there is $\varepsilon > 0$ such that $\mu(X \setminus E) = 1$ for every X-invariant ergodic Borel probability μ with $h(X, \mu) > h(X) - \varepsilon$ and a set is *entropy-full* if its complement is entropy negligible. Systems X and Y are *entropy conjugate* if there is a Borel isomorphism of their actions on entropy full sets ([16]).

Theorem 5.5. *If $\zeta_{X(S,\,S')}(t) = \zeta_{X(T,\,T')}(t)$, then $X(S, S')$ and $X(T, T')$ are entropy conjugate.*

Proof. By Theorem 4.7, $\{\!\{S + S'\}\!\} = \{\!\{T + T'\}\!\}$. So there is a one-to-one correspondence $\alpha : \{\!\{S+S'\}\!\} \to \{\!\{T+T'\}\!\}$. Let E be the set of points in $X(S, S')$ whose starting and ending segments are not 0^∞ or 1^∞ and observe that the set of recurrent points of $X(S, S')$ are in $(X(S, S') \setminus E) \cup \{0^\infty, 1^\infty\}$. Now let $M_1 = \{\mu \in M(X) : \mu(\{0^\infty, 1^\infty\}) \neq 0\}$ and $h_{M_1}(X(S, S')) = \sup_{\mu \in M_1} h_\mu(X(S, S'))$ and choose $0 < \epsilon < \frac{h(X(S,\,S')) - h_{M_1}(X(S,\,S'))}{2}$. Then by definition, E is entropy-negligible. Let $B(S, S') = X(S, S') \setminus E$ and let $B(T, T')$ be the corresponding set for $X(T, T')$.

Define $f : B(S,\, S') \to B(T,\, T')$ such that any word of form $0^{s+1}1^{s'+1}$ maps to $0^{t+1}1^{t'+1}$ where $\alpha(s+s') = t+t'$. Clearly f is a bijection. Also, note that if $\{x_n\}_n$ is a sequence of points in $B(S,\, S')$ mapping to $x \in B(S,\, S')$, then $f(x_n) \to f(x)$ and so f is continuous and so Borel on $B(S, S')$. This proves entropy conjugacy. □

Remark 5.1. Note that the converse of Theorem 5.5 does not hold necessarily. For instance, let $|S|, |S'| = \infty$ with $\{\!\{S+S'\}\!\} = S+S'$ and set $T = \{1\}$ and $T' = S+S'-1$. Then $P_1(X(S,\, S')) = \{0^\infty, 1^\infty\}$ and $P_1(X(T,\, T')) = \{1^\infty\}$. So $\zeta_{X(S,\, S')}(t) \neq \zeta_{X(T,\, T')}(t)$. However, a map f as in the above theorem can be defined.

6 Conclusions

In this work, a generalization of Run-length-limited (RLL) codes, called $(S,\, S')$-gap shifts, has been introduced. This also can be considered as a generalization of an S-gap shift. First, the Fischer cover of an $(S,\, S')$-gap shift is determined in terms of the Fischer covers of two S-gap shifts, one called S-gap shift and the other S'-gap shift. Next, some of its dynamical properties as a subshift has been investigated. Furthermore, its zeta function and entropy is computed. In addition, when $(S,\, S')$-gap shift is SFT, its Bowen-Franks group is obtained. Finally, we give a general necessary condition and some sufficient conditions for conjugacy of $(S,\, S')$-gap shifts.

Competing Interests

The authors declare that no competing interests exist.

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