



## Procedure for Exact Solutions of Nonlinear Pantograph Delay Differential Equations

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### Abstract

This work presents the application of the power series method (PSM) to find solutions of nonlinear delay differential equations of pantograph type (PDDEs). Three equations are solved to show that PSM can provide analytical solutions of PDDEs in convergent series form. The nonlinear pantograph cases study are: a first order equation, a second order equation, and a second order singular equation. Additionally, we present the post-treatment of the power series solutions with the Laplace-Padé (LP) resummation method as a powerful technique to find exact solutions. The proposed methodology possesses a simple procedure based on a few straightforward steps and it does not depend on a perturbation parameter.

*Keywords: Pantograph equations, Power series method, Laplace transform, Padé approximant, Analytical solutions.*

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### 1 Introduction

As widely known, the importance of research on differential equations is that many phenomena, practical or theoretical, can be easily modelled by such equations. In the area of modelling of physical

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phenomenons, we usually assume that the dynamics of the phenomenon under study depends only on the present state as many common phenomena in Physics. Nonetheless, there are problems where this supposition is not true and the application of a traditional model may lead to wrong results. Therefore, it is more accurate to consider that the system possesses memory and its dynamics is influenced by its former state. It is logic to infer that many practical systems exhibit a dynamics that is based on past states of its variables. Such kind of equations are denominated delay differential equations (DDEs).

There are several kinds of delay differential equations; among them, highlights the pantograph equations due to their ability to represent several problems in biology, absorption of light by the interstellar matter, medicine, chemistry, physics, engineering, economics, population studies, number theory, electrodynamics, quantum mechanics, infectious diseases, physiological and pharmaceutical kinetics and chemical kinetics, the navigational control of ships and aircraft and control problems and electronic systems [1, 2]. Its was first reported [3] to study how the electric current is collected by the pantograph of an electric locomotive, from where it gets its name.

In recent years, pantograph equations have received much attention because of their wide application. In [4] was proved the existence and uniqueness of the analytic solution of the multi-pantograph equation. They constructed the Dirichlet series solution and obtained the sufficient condition for the asymptotic stability of the analytic solution obtained. In [5] was addressed the existence and uniqueness of solutions of the pantograph equations and their asymptotic behavior. In [6] presented and proved the theorems, in one-dimensional differential transform method, for solving nonlinear higher order multi-pantograph equations [7].

There exist several works about the numerical and/or analytical solution of such equations: Taylor polynomials [8], differential transform method [9], modified variational iteration Method (VIM) [10, 11], Laplace decomposition algorithm [12, 7], homotopy analysis method (HAM) [13], Picard method [14], collocation method [15] and Runge-Kutta method [16, 17] and Legendre-collocation method [18]. Nevertheless, the power series method (PSM) [19, 20, 21, 22] is a well-known classic straightforward procedure from literature that can be applied successfully to solve differential equations of different kind: linear ordinary differential equations (ODEs) [19, 23, 20, 24], nonlinear ODEs [24, 25, 26, 27, 28, 29], linear partial differential equations (PDEs) [30], generalized pantograph equations [1], among others. This method establishes that the solution of a differential equation can be expressed as a power series of the independent variable.

In this paper we present the application of a hybrid technique combining PSM [31], Laplace Transform (LT) and Padé Approximant (PA) [32] to find analytical solutions for pantograph delay differential equations (PDDEs) [33, 34, 35, 36, 37, 38, 39, 40]. Solutions to PDDEs are first obtained in convergent series form using the PSM. To improve the solution obtained from PSM's truncated series, we apply LT to it, then convert the transformed series into a meromorphic function by forming its PA. Finally, we take the inverse LT of the PA to obtain the analytical solution. This hybrid method (LPPSM), which combines PSM with Laplace-Padé resummation greatly improves PSM's truncated series solutions in convergence rate. In fact, the Laplace-Padé resummation method enlarges the domain of convergence of the truncated power series and often leads to the exact solution.

It is important to highlight that the proposed method does not produce secular terms (noise terms) as the homotopy perturbation based techniques [41]. This reduces the volume of computation and

improves the efficiency of the method in comparison to the perturbation based methods. Thereupon, LPPSM does not require a perturbation parameter as the perturbation based techniques including homotopy perturbation method (HPM). Finally, LPPSM is straightforward and can be programmed using computer algebra packages like Maple or Mathematica.

The rest of this paper is organized as follows. In the next section we illustrate the basic concept of the PSM. The main idea behind the Padé approximant is given in section 3. In section 4, we give the basic concept of the Laplace-Padé resummation method. The application of PSM to solve pantograph equations is depicted in section 5. In section 6, we apply LPPSM to solve three PDDEs. In section 7, we give a brief discussion. Finally, a conclusion is drawn in the last section.

## 2 Basic Concept of Power Series Method

It can be considered that a nonlinear differential equation can be expressed as

$$A(u) - f(t) = 0, \quad t \in \Omega, \quad (2.1)$$

having as boundary condition

$$B(u, \partial u / \partial \eta) = 0, \quad t \in \Gamma, \quad (2.2)$$

where  $A$  is a general differential operator,  $f(t)$  is a known analytic function,  $B$  is a boundary operator, and  $\Gamma$  is the boundary of domain  $\Omega$ .

PSM [19, 20] establishes that the solution of a differential equation can be written as

$$u(t) = \sum_{n=0}^{\infty} u_n t^n, \quad (2.3)$$

where  $u_0, u_1, \dots$  are unknowns to be determined by series method.

The basic process of series method can be described as:

1. Equation (2.3) is substituted into (2.1), then we regroup the equation in terms of powers of  $t$ .
2. We equate all coefficients of powers of  $t$  to zero in the resulting polynomial.
3. The boundary conditions of (2.1) are substituted into (2.3) to generate an algebraic equation for each boundary condition.
4. Aforementioned steps generate algebraic equations for the unknowns of (2.3).
5. Finally, we solve the algebraic equations to obtain the coefficients  $u_0, u_1, \dots$

### 3 Padé Approximant

Given an analytical function  $u(t)$  with Maclaurin's expansion

$$u(t) = \sum_{n=0}^{\infty} u_n t^n, \quad 0 \leq t \leq T. \quad (3.1)$$

The Padé approximant to  $u(t)$  of order  $[L, M]$  which we denote by  $[L/M]_u(t)$  is defined by [32]

$$[L/M]_u(t) = \frac{p_0 + p_1 t + \dots + p_L t^L}{1 + q_1 t + \dots + q_M t^M}, \quad (3.2)$$

where we considered  $q_0 = 1$ , and the numerator and denominator have no common factors.

The numerator and the denominator in (3.2) are constructed so that  $u(t)$  and  $[L/M]_u(t)$  and their derivatives agree at  $t = 0$  up to  $L + M$ . That is

$$u(t) - [L/M]_u(t) = O(t^{L+M+1}). \quad (3.3)$$

From (3.3), we have

$$u(t) \sum_{n=0}^M q_n t^n - \sum_{n=0}^L p_n t^n = O(t^{L+M+1}). \quad (3.4)$$

From (3.4), we get the following algebraic linear systems

$$\begin{cases} u_L q_1 + \dots + u_{L-M+1} q_M = -u_{L+1} \\ u_{L+1} q_1 + \dots + u_{L-M+2} q_M = -u_{L+2} \\ \vdots \\ u_{L+M-1} q_1 + \dots + u_L q_M = -u_{L+M}, \end{cases} \quad (3.5)$$

and

$$\begin{cases} p_0 = u_0 \\ p_1 = u_1 + u_0 q_1 \\ \vdots \\ p_L = u_L + u_{L-1} q_1 + \dots + u_0 q_L. \end{cases} \quad (3.6)$$

From (3.5), we calculate first all the coefficients  $q_n$ ,  $1 \leq n \leq M$ . Then, we determine the coefficient  $p_n$ ,  $0 \leq n \leq L$  from (3.6).

Note that for a fixed value of  $L + M + 1$ , the error (3.3) is smallest when the numerator and denominator of (3.2) have the same degree or when the numerator has degree one higher than the denominator.

### 4 Laplace-Padé Resummation Method

Several approximate methods provide power series solutions (polynomial). Nevertheless, sometimes, this type of solutions lacks of large domains of convergence. Therefore, Laplace-Padé [33, 34, 35, 36, 37, 38, 39, 40] resummation method is used in literature to enlarge the domain of convergence of solutions or inclusive to find exact solutions.

The Laplace-Padé Method can be Explained as Follows:

1. First, Laplace  $s$  transformation is applied to the  $t$  power series (2.3), where  $s$  and  $t$  take values in the complex plane and real line respectively.
2. Next,  $s$  is substituted by  $1/t$  in the resulting equation.
3. After that, we convert the transformed series into a meromorphic function by forming its Padé approximant of order  $[N/M]$ .  $N$  and  $M$  are arbitrarily chosen, but they should be of smaller value than the order of the power series. In this step, the Padé approximant extends the domain of the truncated series solution to obtain better accuracy and convergence.
4. Then,  $t$  is substituted by  $1/s$ .
5. Finally, by using the inverse Laplace  $s$  transformation, we obtain the exact or approximate solution.

## 5 Application of PSM to Solve Pantograph Equations

Many application problems in science and engineering lead to the solution of pantograph equations of the form

$$u^{(n)}(t) = f(t, u(t), \dots, u^{(n-1)}(t), u(pt), \dots, u^{(n-1)}(pt)), \quad (5.1)$$

$$u^{(k)}(0) = \eta_k, k = 0, \dots, n - 1, \quad (5.2)$$

where  $\eta_k, k = 0, \dots, n - 1$  are given and  $0 < p < 1$ .

We assume that the solution to initial value problem (5.1)-(5.2) exists, is unique and analytic. To simplify the exposition of the PSM, we integrate equation (5.1)  $n$  times with respect to  $t$  and use the initial conditions (5.2) to obtain

$$u(t) = \sum_{k=0}^{n-1} \frac{\eta_k}{k!} t^k + \int_0^t \dots \int_0^t f(t, u(t), \dots, u^{(n-1)}(t), u(pt), \dots, u^{(n-1)}(pt)) dt \dots dt. \quad (5.3)$$

It is important to note that the preprocessing step (5.3) is not relevant to the solution procedure presented here, so one can apply the PSM directly to (5.1).

In view of PSM, we assume the solution  $u(t)$ , to have the form

$$u(t) = u_0 + u_1 t + u_2 t^2 + \dots, \quad (5.4)$$

where  $u_0, u_1, u_2, \dots$  are unknown coefficients to be determined later on by the PSM.

Then substitute (5.4) into equation (5.3) and equate the coefficients of powers of  $t$  to zero in the resulting polynomial equation to get a recursion equation for these coefficients. Finally, we use equation (5.4) to obtain the exact solution as power series.

The solutions series obtained from PSM may have limited regions of convergence, even if we take a large number of terms. Therefore, we apply the Laplace-Padé post-treatment to PSM's truncated series which we call LPPSM to enlarge the convergence region as depicted in the next section.

## 6 Cases Study

In this section, we will demonstrate the effectiveness and accuracy of the LPPSM presented in the previous section through three PDDEs.

### 6.1 First order nonlinear pantograph equation

Consider the following nonlinear pantograph equation

$$u'(t) = 2 - u^2(t/2), t \geq 0, \quad (6.1)$$

$$u(0) = 0. \quad (6.2)$$

The exact solution of initial value problem (6.1)-(6.2) is

$$u(t) = 2 \sin t. \quad (6.3)$$

In order to simplify the exposition of the LPPSM presented in section 4 to solve (6.1)-(6.2), we first integrate equation (6.1) once with respect to  $t$  and use the initial condition (6.2) to get

$$u(t) = u(0) + \int_0^t 2 - u^2(t/2) dt. \quad (6.4)$$

In view of the PSM, we assume that the solution  $u(t)$  has the form

$$u(t) = u_0 + u_1 t + u_2 t^2 + \dots, \quad (6.5)$$

where  $u_n, n = 0, 1, 2, \dots$  are unknown coefficients to be determined later on by the PSM.

Then, we substitute (6.5) into (6.4) and use the initial condition to get

$$\sum_{n=0}^{\infty} u_n t^n - \int_0^t 2 - \left( \sum_{n=0}^{\infty} \left( \frac{u_n}{2^n} \right) t^n \right)^2 dt = 0. \quad (6.6)$$

This yields

$$2u_0 + (u_1 - 2)t + \sum_{n=2}^{\infty} \left( u_n + \frac{1}{2^{n-1}n} \sum_{k=0}^{n-1} u_k u_{n-1-k} \right) t^n = 0. \quad (6.7)$$

Equating the coefficients of powers of  $t$  to zero in (6.7), we have

$$u_0 = 0, u_1 = 2,$$

and the following recursion for the unknown coefficients  $u_n$

$$u_n = -\frac{1}{2^{n-1}n} \sum_{k=0}^{n-1} u_k u_{n-1-k}, n = 2, 3, \dots \quad (6.8)$$

From this recursion, we compute some coefficients

$$u_2 = 0, u_3 = -1/3, u_4 = 0, u_5 = 1/60, u_6 = 0, u_7 = -1/2520.$$

Then using (6.5) and the coefficients above, we obtain

$$u(t) = 2t - \frac{t^3}{3} + \frac{t^5}{60} - \frac{t^7}{2520}. \quad (6.9)$$

The solutions series obtained from the PSM may have limited regions of convergence, even if we take a large number of terms. Accuracy can be increased by applying the Laplace-Padé post-treatment. First, we apply  $t$ -Laplace transform to (6.9). Then, we substitute  $s$  by  $1/t$  and apply  $t$ -Padé approximant to the transformed series. Finally, we substitute  $t$  by  $1/s$  and apply the inverse Laplace  $s$ -transform to the resulting expression to obtain the approximate solution or exact solution.

Applying Laplace transform to (6.9) yields

$$\mathcal{L}[u(t)] = \frac{2}{s^2} - \frac{2}{s^4} + \frac{2}{s^6} - \frac{2}{s^8}. \tag{6.10}$$

For the sake of simplicity we let  $s = 1/t$ , then

$$\mathcal{L}[u(t)] = 2t^2 - 2t^4 + 2t^6 - 2t^8. \tag{6.11}$$

All of the  $[L/M]$   $t$ -Padé approximants of (6.11) with  $L \geq 1$  and  $M \geq 1$  and  $L + M \leq 8$  yield

$$[L/M]_u = \frac{2t^2}{1+t^2}. \tag{6.12}$$

Now since  $t = 1/s$ , we obtain  $[L/M]_u$  in terms of  $s$  as follows

$$[L/M]_u = \frac{2}{1+s^2}. \tag{6.13}$$

Finally, applying the inverse LT to the Padé approximants (6.13), we obtain the approximate solution which is in this case the exact solution (6.3) in closed form.

## 6.2 Second order nonlinear pantograph equation

Consider the following nonlinear pantograph equation

$$u''(t) = -u(t) + 5u^2(t/2), \quad t \geq 0, \tag{6.14}$$

$$u(0) = 1, \quad u'(0) = -2. \tag{6.15}$$

The exact solution of initial value problem (6.14)-(6.15) is

$$u(t) = e^{-2t}. \tag{6.16}$$

In order to simplify the exposition of the LPPSM presented in section 4 to solve (6.14)-(6.15), we first integrate equation (6.14) twice with respect to  $t$  and use the initial condition (6.15) to get

$$u(t) = u(0) + u'(0)t + \int_0^t \int_0^t -u(t) + 5u^2(t/2) dt dt. \tag{6.17}$$

In view of the PSM, we assume that the solution  $u(t)$  has the form

$$u(t) = u_0 + u_1t + u_2t^2 + \dots, \tag{6.18}$$

where  $u_n, n = 0, 1, 2, \dots$  are unknown coefficients to be determined later on by the PSM.

Then, we substitute (6.18) into equation (6.17) to get

$$\sum_{n=0}^{\infty} u_n t^n - 1 + 2t + \int_0^t \int_0^t \sum_{n=0}^{\infty} u_n t^n - 5 \left( \sum_{n=0}^{\infty} \left( \frac{u_n}{2^n} \right) t^n \right)^2 dt dt = 0. \tag{6.19}$$

This yields

$$(u_0 - 1) + (u_1 + 2)t + \sum_{n=2}^{\infty} \left( u_n + \frac{u_{n-2}}{n(n-1)} - \frac{5}{2^{n-2}n(n-1)} \sum_{k=0}^{n-2} u_k u_{n-2-k} \right) t^n = 0. \quad (6.20)$$

Equating the coefficients of powers of  $t$  to zero in (6.20), we have

$$u_0 = 1, u_1 = -2,$$

and the following recursion for the unknown coefficients  $u_n$

$$u_n = -\frac{1}{n(n-1)} \left( u_{n-2} - \frac{5}{2^{n-2}} \sum_{k=0}^{n-2} u_k u_{n-2-k} \right), n = 2, 3, \dots \quad (6.21)$$

From this recursion, we compute some coefficients

$$u_2 = 2, u_3 = -4/3, u_4 = 2/3, u_5 = -4/15, u_6 = 4/45.$$

Then using (6.18) and the coefficients above, we obtain

$$u(t) = 1 - 2t + 2t^2 - \frac{4}{3}t^3 + \frac{2}{3}t^4 - \frac{4}{15}t^5 + \frac{4}{45}t^6. \quad (6.22)$$

The solutions series obtained from the PSM may have limited regions of convergence, even if we take a large number of terms. Accuracy can be increased by applying the Laplace-Padé post-treatment. First, we apply  $t$ -Laplace transform to (6.22). Then, we substitute  $s$  by  $1/t$  and apply  $t$ -Padé approximant to the transformed series. Finally, we substitute  $t$  by  $1/s$  and apply the inverse Laplace  $s$ -transform to the resulting expression to obtain the approximate solution or exact solution.

Applying Laplace transform to (6.22) yields

$$\mathcal{L}[u(t)] = \frac{1}{s} - \frac{2}{s^2} + \frac{4}{s^3} - \frac{8}{s^4} + \frac{16}{s^5} - \frac{32}{s^6} + \frac{64}{s^7}. \quad (6.23)$$

For the sake of simplicity we let  $s = 1/t$ , then

$$\mathcal{L}[u(t)] = t - 2t^2 + 4t^3 - 8t^4 + 16t^5 - 32t^6 + 64t^7. \quad (6.24)$$

All of the  $[L/M]$   $t$ -Padé approximants of (6.24) with  $L \geq 1$  and  $M \geq 1$  and  $L + M \leq 7$  yield

$$[L/M]_u = \frac{t}{1 + 2t}. \quad (6.25)$$

Now since  $t = 1/s$ , we obtain  $[L/M]_u$  in terms of  $s$  as follows

$$[L/M]_u = \frac{1}{s + 2}. \quad (6.26)$$

Finally, applying the inverse LT to the Padé approximants (6.26), we obtain the approximate solution which is in this case the exact solution (6.16) in closed form.

### 6.3 Second order singular pantograph equation

Consider the following nonlinear pantograph equation

$$u''(t) = u(t) - \frac{8}{t^2} u^2(t/2), t \geq 0, \quad (6.27)$$

$$u(0) = 0, u'(0) = 1. \quad (6.28)$$



The exact solution of initial value problem (6.27)-(6.28) is

$$u(t) = te^{-t}. \tag{6.29}$$

In order to simplify the exposition of the LPPSM presented in section 4 to solve (6.27)-(6.28), we first integrate equation (6.27) twice with respect to  $t$  and use the initial condition (6.28) to get

$$u(t) = u(0) + u'(0)t + \int_0^t \int_0^t u(t) - \frac{8}{t^2} u^2(t/2) dt dt. \tag{6.30}$$

In view of the PSM, we assume that the solution  $u(t)$  has the form

$$u(t) = u_0 + u_1t + u_2t^2 + \dots, \tag{6.31}$$

where  $u_n, n = 0, 1, 2, \dots$  are unknown coefficients to be determined later on by the PSM.

Then, we substitute (6.31) into equation (6.30) to get

$$\sum_{n=0}^{\infty} u_n t^n - t - \int_0^t \int_0^t \sum_{n=0}^{\infty} u_n t^n - \frac{8}{t^2} \left( \sum_{n=0}^{\infty} \left( \frac{u_n}{2^n} \right) t^n \right)^2 dt dt = 0. \tag{6.32}$$

This yields

$$u_0 + (u_1 - 1)t + \sum_{n=2}^{\infty} \left( u_n - \frac{u_{n-2}}{n(n-1)} + \frac{8}{2^n n(n-1)} \sum_{k=1}^{n-1} u_k u_{n-k} \right) t^n = 0. \tag{6.33}$$

Equating the coefficients of powers of  $t$  to zero in (6.33), we have

$$u_0 = 0, u_1 = 1,$$

and the following recursion for the unknown coefficients  $u_n$

$$u_n = \frac{1}{n(n-1)} \left( u_{n-2} - \frac{8}{2^n} \sum_{k=0}^{n-1} u_k u_{n-k} \right), n = 2, 3, \dots \tag{6.34}$$

From this recursion, we compute some coefficients

$$u_2 = -1, u_3 = 1/2, u_4 = -1/3!, u_5 = 1/4!, u_6 = -1/5!.$$

Then using (6.31) and the coefficients above, we obtain

$$u(t) = t - t^2 + \frac{1}{2}t^3 - \frac{1}{3!}t^4 + \frac{1}{4!}t^5 - \frac{1}{5!}t^6. \tag{6.35}$$

The solutions series obtained from the PSM may have limited regions of convergence, even if we take a large number of terms. Accuracy can be increased by applying the Laplace-Padé post-treatment. First, we apply  $t$ -Laplace transform to (6.35). Then, we substitute  $s$  by  $1/t$  and apply  $t$ -Padé approximant to the transformed series. Finally, we substitute  $t$  by  $1/s$  and apply the inverse Laplace  $s$ -transform to the resulting expression to obtain the approximate solution or exact solution.

Applying Laplace transform to (6.35) yields

$$\mathcal{L}[u(t)] = \frac{1}{s^2} - \frac{2}{s^3} + \frac{3}{s^4} - \frac{4}{s^5} + \frac{5}{s^6} - \frac{6}{s^7}. \tag{6.36}$$

For the sake of simplicity we let  $s = 1/t$ , then

$$\mathcal{L}[u(t)] = t^2 - 2t^3 + 3t^4 - 4t^5 + 5t^6 - 6t^7. \tag{6.37}$$

All of the  $[L/M]$   $t$ -Padé approximants of (6.37) with  $L \geq 1$  and  $M \geq 1$  and  $L + M \leq 7$  yield

$$[L/M]_u = \frac{t^2}{t^2 + 2t + 1}. \quad (6.38)$$

Now since  $t = 1/s$ , we obtain  $[L/M]_u$  in terms of  $s$  as follows

$$[L/M]_u = \frac{1}{s^2 + 2s + 1}. \quad (6.39)$$

Finally, applying the inverse LT to the Padé approximants (6.39), we obtain the approximate solution which is in this case the exact solution (6.29) in closed form.

## 7 Discussion

In this paper we presented the power series method (PSM) as a useful analytical tool to solve PDDEs. Three PDDEs problems were solved by this method leading to the exact solutions. For each one of the problems solved here, the PSM transformed the PDDE into an easily solvable linear equation for the coefficient of the power series solution. To improve the PSM solution, a Laplace-Padé (LP) resummation is applied to the PSM's truncated series leading to the exact solution. Additionally, the solution procedure does not involve unnecessary computation like that related to noise terms [41]. This reduces the volume of computation and improves the efficiency of the method. It is important to notice that the high complexity of these problems was effectively handled by LPPSM method due to the power of PSM and resummation capability of Laplace-Padé.

On one side, semi-analytic methods like HPM [42, 43, 44, 45, 46, 47, 48, 49], HAM, VIM among others, require an initial approximation for the sought solutions and the computation of one or several adjustment parameters. If the initial approximation is properly chosen the results can be highly accurate, nonetheless, no general methods are available to choose such initial approximation. This issue motivates the use of adjustment parameters obtained by minimizing the least-squares error with respect to the numerical solution.

On the other side, PSM or LPPSM methods do not require any trial equation as requisite for the starting the method. What is more, PSM obtains its coefficients using an easy computable straightforward procedure that can be implemented into programs like Maple or Mathematica. Finally, if the solution of the PDDE is not expressible in terms of known functions then the LP resummation will provide a larger domain of convergence.

## 8 Conclusions

This work presented LPPSM method as a combination of the classic PSM and a resummation method based on the Laplace transforms and Padé approximant. Firstly, the solutions of PDDEs are obtained in convergent series forms using PSM. Next, in order to enlarge the domain of convergence of the truncated power series, a post-treatment combining Laplace transform and Padé approximant is applied. This technique denominate LPPSM improves PSM's truncated series solutions in convergence rate, and often leads to the exact solution. What is more, PSM is a powerful tool, because it does

not require of a perturbation parameter to work and it does not generate secular terms as other semi-analytical methods like HPM, HAM or VIM.

By solving three problems, we presented the LPPSM as a handy tool with high potential to solve linear/nonlinear PDDEs. Furthermore, we obtained successfully the exact solutions of such three problems highlighting the efficiency of LPPSM. In addition, the proposed method is based on a straightforward procedure, suitable for engineers. Finally, further work should be performed to solve other nonlinear PDDEs systems.

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## Competing Interests

The authors declare that no competing interests exist.

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