British Journal of Mathematics & Computer Science
10(1): 1-8, 2015, Article no.BJMCS.17068
ISSN: 2231-0851

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A Refinement of Jensen's Inequality via Superquadratic Functions

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Article Information
DOI: 10.9734/BJMCS/2015/17068
Editor(s):
(1) Dijana Mosic, Department of Mathematics, University of Nis, Serbia.
Reviewers:
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Complete Peer review History:http://sciencedomain.org/review-history/9874

Original Research Article

Received: 25 February 2015
Accepted: 17 April 2015
Published: 20 June 2015

Abstract

We establish the inequality
$\varphi(\frac{1}{2^n}a + \frac{2^n-1}{2^n}b) \leq \frac{1}{2^n}\varphi(a) + \frac{2^n-1}{2^n}\varphi(b) - [\sum_{i=1}^{n}(2^{i-n})\varphi(|\frac{b-a}{2^i}|)]$, where φ is superquadratic and prove a more general inequality where $\frac{1}{2^n}$ and $\frac{2^n-1}{2^n}$ are replaced by $\frac{\alpha}{2^n}$ and $\frac{\beta}{2^n}$ respectively, with $\alpha + \beta = 2^n,\ n \in \mathbb{N}$, the latter is extended to the case where $\alpha + \beta \neq 2^n,\ n \in \mathbb{N}$.

Keywords: Convex function; Midconvex function; Superquadratic function; Jensen's inequality and Refined Jensen's inequality.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

1 Introduction

The discrete version of Jensen's inequality for functions is given in [1,p 1] as

$$\phi(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \phi(x_i), \tag{1.1}$$

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where ϕ is a convex function defined on an interval I in $\mathbb{R}$, $(x_1, ..., x_n) \in I^n (n \geq 2)$ and $(p_1, ..., p_n)$ is any non-negative n-tuple satisfying $P_n = \sum_{i=1}^{n} p_i > 0$.

Definition 1.1. [2] A function $\varphi : [0, \infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $P_x \in \mathbb{R}$ such that

$$\varphi(y) \geq \varphi(x) + (y - x)P_x + \varphi(|y - x|) \tag{1.2}$$

for all y ≥ 0.

By replacing the convex function with superquadratic function in (1.1) [1, Lemma A], we obtain;

$$\varphi\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi\Big(\Big|x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\Big|\Big), \tag{1.3}$$

for $x_i, p_i \geq 0, i = 1, ..., n$ and $P_n = \sum_{i=1}^{n} p_i > 0$.
The equivalent continuous version of (1.3) is given in [2] as

$$\varphi\Big(\int f d\mu\Big) \leq \int [\varphi(f(s)) - \varphi(|f(s) - \textstyle\int f d\mu|)] d\mu(s), \tag{1.4}$$

for all probability measures μ and all non-negative, μ-integrable functions f.
The inequality (1.4) is referred to as the refined continuous Jensen's inequality.
In [2, Theorem 2.3], the inequality (1.4) was proved to be equivalent to definition (1.1) and in [3, Theorem 9] the equivalence between (1.3) and definition (1.1) was also established.
We refer the reader to [2,4,5,6] for general properties and some applications of superquadratic functions .

Definition 1.2. [4] A function $\varphi : X \to \mathbb{R}$ is said to be superquadratic,
if $\forall x, y \in X$

$$\varphi(x + y) + \varphi(x - y) \geq 2[\varphi(x) + \varphi(y)], \tag{1.5}$$

where X is a real vector space.

With the change of variable, $x - y = a$ and $x + y = b$, (1.5) becomes

$$\varphi\Big(\frac{a + b}{2}\Big) \leq \frac{1}{2}\varphi(a) + \frac{1}{2}\varphi(b) - \varphi\Big(\frac{b - a}{2}\Big). \tag{1.6}$$

From inequality (1.3) when we set $n = 2$ and $p_1 = p_2 = 1$, we obtain

$$\varphi\Big(\frac{x_1 + x_2}{2}\Big) \leq \frac{1}{2}\varphi(x_1) + \frac{1}{2}\varphi(x_2) - \varphi\Big(\Big|\frac{x_1 - x_2}{2}\Big|\Big). \tag{1.7}$$

From (1.7) one can clearly see that all functions satisfying definition (1.1) also satisfies definition (1.2) but the converse is not always true even if we restrict the domain of φ in definition (1.2) to $\mathbb{R}^+$. For examples see [4].
We refer the reader to [7] for some general properties of superquadratic functions according to definition (1.2).

Definition 1.3. [8. p 211] Let L be a normed vector space and $U \subseteq L$ be a convex set. A function $f : U \to \mathbb{R}$ is said to be midconvex if $\forall x, y \in U$

$$f\Big(\frac{x + y}{2}\Big) \leq \frac{1}{2}[f(x) + f(y)]. \tag{1.8}$$

An equivalent definition is given for midconvexity using n points, this definition depends on the notion of rational convex combination of points. We give this equivalent definition for midconvexity as a theorem.

Theorem 1.1. *[8, p 212]. f is midconvex on the convex set $U \subseteq L$ if and only if for any rational convex combination of points in U*

$$f(\sum_{i=1}^{n} \alpha_i x_i) \leq \sum_{i=1}^{n} \alpha_i f(x_i), \tag{1.9}$$

where $\alpha_i \geq 0$ for $i = 1, ..., n$, $\alpha_i' s$ are rational and $\sum_{i=1}^{n} \alpha_i = 1$.

convex functions are continuous on the interior of its domain, while midconvex functions are not necessarilly continuous and it is clear that convexity of a function implies midconvexity.

Theorem 1.2. *[9,Theorem 1.1.4]. Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if and only if f is midconvex, where I is a nondegenerate interval.*

2 Main Results

In this paper, we prove a new refinement of the Jensen's inequality by using definition (1.2) of superquadraticity, where we allow φ to be midconvex but not convex and φ is from $[0, \infty)$ to $[0, \infty)$.

Proposition 2.1. *For $\lambda_1 = \frac{1}{2^n}, \lambda_2 = \frac{2^n-1}{2^n}$ and for all $n \in \mathbb{N}$*

$$\varphi(\lambda_1 a + \lambda_2 b) \leq \lambda_1 \varphi(a) + \lambda_2 \varphi(b) - [\sum_{i=1}^{n} (\lambda_1 2^i) \varphi(|\frac{b-a}{2^i}|)] \tag{2.1}$$

where φ satisfies definition (1.2).

Proof

We establish (2.1) using induction.
For $n = 1$, (2.1) is (1.6).
Fix $k \in \mathbb{N}, k > 1$ and suppose (2.1) is true, that is
$\varphi(\frac{a}{2^k} + \frac{2^k-1}{2^k} b) \leq \frac{1}{2^k}\varphi(a) + \frac{2^k-1}{2^k}\varphi(b) - [\sum_{i=1}^{k} (\frac{1}{2^{k-i}}) \varphi(|\frac{b-a}{2^i}|)]$.
Now

$$\begin{aligned} \varphi(\frac{a}{2^{k+1}} + \frac{2^{k+1}-1}{2^{k+1}} b) &= \varphi(\frac{1}{2}(\frac{a}{2^k} + \frac{2^{k+1}-1}{2^k} b)) \\ &= \varphi(\frac{1}{2}(\frac{a + (2^k-1)b}{2^k} + b)) \end{aligned}$$

and so by inequality (1.6),

$$\begin{aligned} \varphi(\frac{1}{2}[\frac{a + (2^k-1)b}{2^k} + b]) &\leq \frac{1}{2}\varphi(\frac{a + (2^k-1)b}{2^k}) + \frac{1}{2}\varphi(b) - \varphi(\frac{1}{2}|\frac{a + (2^k-1)b}{2^k} - b|) \\ &= \frac{1}{2}\varphi(\frac{a + (2^k-1)b}{2^k}) + \frac{1}{2}\varphi(b) - \varphi(|\frac{b-a}{2^{k+1}}|). \end{aligned}$$

But $\varphi(\frac{a}{2^k} + \frac{2^k-1}{2^k} b) \leq \frac{1}{2^k}\varphi(a) + \frac{2^k-1}{2^k}\varphi(b) - [\ \sum_{i=1}^{k} (\frac{1}{2^{k-i}}) \varphi(|\frac{b-a}{2^i}|)]$, from the inductive hypothesis.
So $\varphi\big(\frac{a}{2^{k+1}} + \frac{2^{k+1}-1}{2^{k+1}} b\big)$
$\leq \frac{1}{2^{k+1}}\varphi(a) + \frac{2^k-1}{2^{k+1}}\varphi(b) - \frac{1}{2}\big[\sum_{i=1}^{k} \big(\frac{1}{2^{k-i}}\big) \varphi\big(\big|\frac{b-a}{2^i}\big|\big)\big] + \frac{1}{2}\varphi(b) - \varphi(|\frac{b-a}{2^{k+1}}|)$.

Thus

$$\varphi(\frac{a}{2^{k+1}} + \frac{2^{k+1}-1}{2^{k+1}} b) \leq \frac{1}{2^{k+1}}\varphi(a) + \frac{2^{k+1}-1}{2^{k+1}}\varphi(b) - [\sum_{i=1}^{k+1} (\frac{1}{2^{k+1-i}}) \varphi(|\frac{b-a}{2^i}|)],$$

which completes the proof.

Proposition 2.2.

$$\varphi(\frac{\alpha}{2^n}a+\frac{\beta}{2^n}b) \tag{2.2}$$
$$\leq \frac{\alpha}{2^n}\varphi(a)+\frac{\beta}{2^n}\varphi(b)-\Big[\sum_{i=1}^{2}\big(\frac{1}{2^{n-i}}\big)\varphi(|\frac{b-a}{2^i}|)+\sum_{i=3}^{n}\big(\frac{1}{2^{n-i}}\big)\varphi(\frac{\epsilon_{(n+1-i)}}{2^i}|b-a|)\Big],$$

where $\alpha,\beta\in\mathbb{N}$ such that $\alpha+\beta=2^n$, $n-2$, is the number of splits and $\epsilon_{(n+1-i)}$ is the minimum of $\{\alpha,\ \beta\}$ before the $(n+1-i)$ split.

Definition 2.1. For $\lambda_1=\frac{\alpha}{2^n},\lambda_2=\frac{\beta}{2^n}$ we define a split of $\varphi(\frac{\alpha}{2^n}a+\frac{\beta}{2^n})$ to be

$$\varphi(\frac{1}{2}[\frac{\alpha a+(\beta-2^{n-1})b}{2^{n-1}}+b]).$$

To demonstrate the techniques in the proof of proposition (2.2), we first consider some examples for given values of n,α and β.

Example 2.1. *We consider the case where $n=3,\ \alpha=3$ and $\beta=5$, that is*

$$\begin{aligned}\varphi(\frac{3a}{8}+\frac{5b}{8}) &= \varphi(\frac{1}{2}[\frac{3a}{4}+\frac{5b}{4}])\\ &= \varphi(\frac{1}{2}[(\frac{3a}{4}+\frac{b}{4})+b])\\ &\leq \frac{1}{2}\varphi(\frac{3a+b}{4})+\frac{1}{2}\varphi(b)-\varphi(\frac{3}{8}|b-a|),\end{aligned}$$

using the split and (1.6).
From inequality (2.1)

$$\varphi(\frac{3a}{4}+\frac{b}{4})\leq\frac{3}{4}\varphi(a)+\frac{1}{4}\varphi(b)-[\frac{1}{2}\varphi(|\frac{b-a}{2}|)+\varphi(|\frac{b-a}{4}|)].$$

So

$$\varphi(\frac{3a}{8}+\frac{5b}{8})\leq\frac{3}{8}\varphi(a)+\frac{1}{8}\varphi(b)+\frac{1}{2}\varphi(b)-\frac{1}{2}[\frac{1}{2}\varphi(|\frac{b-a}{2}|)+\varphi(|\frac{b-a}{4}|)]-\varphi(\frac{3}{8}|b-a|)$$
$$\varphi(\frac{3a}{8}+\frac{5b}{8})\leq\frac{3}{8}\varphi(a)+\frac{5}{8}\varphi(b)-[\frac{1}{4}\varphi(|\frac{b-a}{2}|)+\frac{1}{2}\varphi(|\frac{b-a}{4}|)+\varphi(\frac{3}{8}|b-a|)]$$

Example 2.2. *We consider the case where $n=4,\ \alpha=5$ and $\beta=11$, that is*

$$\begin{aligned}\varphi(\frac{5a}{16}+\frac{11b}{16}) &= \varphi(\frac{1}{2}[\frac{5a}{8}+\frac{11b}{8}])\\ &= \varphi(\frac{1}{2}[(\frac{5a+3b}{8}+b])\\ &\leq \frac{1}{2}\varphi(\frac{5a+3b}{8})+\frac{1}{2}\varphi(b)-\varphi(\frac{5}{16}|b-a|),\end{aligned}$$

using the split and (1.6).

Now from example (2.1),

$$\varphi(\frac{5a+3b}{8})\leq\frac{5}{8}\varphi(a)+\frac{3}{8}\varphi(b)-[\frac{1}{4}\varphi(|\frac{b-a}{2}|)+\frac{1}{2}\varphi(|\frac{b-a}{4}|)+\varphi(\frac{3}{8}|b-a|)].$$

So

$$\begin{aligned}\varphi\big(\frac{5a}{16}+\frac{11b}{16}\big) &\leq \frac{5}{16}\varphi(a)+\frac{3}{16}\varphi(b)-\big[\frac{1}{8}\varphi\big(\big|\frac{b-a}{2}\big|\big)+\frac{1}{4}\varphi\big(\big|\frac{b-a}{4}\big|\big)\big] \\ &- \frac{1}{2}\varphi\big(\frac{3}{8}|b-a|\big)+\frac{1}{2}\varphi(b)-\varphi\big(\frac{5}{16}|b-a|\big)\end{aligned}$$

This simplifies to

$\varphi\big(\frac{5a}{16}+\frac{11b}{16}\big)$

$$\leq \frac{5}{16}\varphi(a)+\frac{11}{16}\varphi(b)-\big[\frac{1}{8}\varphi\big(\big|\frac{b-a}{2}\big|\big)+\frac{1}{4}\varphi\big(\big|\frac{b-a}{4}\big|\big)\big]-\big[\frac{1}{2}\varphi\big(\frac{3}{8}|b-a|\big)+\varphi\big(\frac{5}{16}|b-a|\big)\big].$$

Proof of proposition (2.2)

If α and β are both even, say $2\delta=\alpha$ and $2\sigma=\beta$ where $\delta,\ \ \sigma\ \in\mathbb{Z}^+$, then $\delta+\sigma=2^{n-1}$ and so we only need to prove the result for α,β both odd.
Let $\alpha=2\delta+1$ and $\beta=2\sigma+1$, where $\delta,\sigma\in\mathbb{Z}^+\cup\{0\}$, so
$2(\delta+\sigma+1)=2^n$ and therefore $\sigma=2^{n-1}-\delta-1$.
Without loss of generality let $\sigma\geq\delta$.
We consider the two special cases, $n=1$ and $n=2$.
Case 1 where $n=1$, we have $\delta=\sigma=0$ and $\alpha=\beta=1$, $(n-2)$ is -1. But we cannot have a negative number of splits, so we ignore the last expression in (2.2) and summing i from 1 to 1 since $n=1$, (2.2) reduces to

$$\varphi\big(\frac{1}{2}a+\frac{1}{2}b\big)\leq\frac{1}{2}\varphi(a)+\frac{1}{2}\varphi(b)-\varphi\big(\big|\frac{b-a}{2}\big|\big)\big],$$

which is precisely inequality (1.6).

Case 2 where $n=2$, the number of splits is zero and we have only two odd numbers 1 and 3, thus we have the expression $\varphi\big(\frac{a+3b}{4}\big)$ to expand.

This is (2.1) a specal case of (2.2) for $\lambda_1=\frac{1}{4}$ and $\lambda_2=\frac{3}{4}$, giving

$$\varphi\big(\frac{a+3b}{4}\big)\leq\frac{1}{4}\varphi(a)+\frac{3}{4}\varphi(b)-\big[\frac{1}{2}\varphi\big(\frac{1}{2}|b-a|\big)+\varphi\big(\frac{1}{4}|b-a|\big)\big]$$

$$=\frac{1}{4}\varphi(a)+\frac{3}{4}\varphi(b)-\big[\sum_{i=1}^{2}\big(\frac{1}{2^{2-i}}\big)\varphi\big(\big|\frac{b-a}{2^i}\big|\big)\big],$$

hence inequality (2.2) is true for the special cases where $n=1$ and $n=2$.

We consider $n=2$ as our base point since the number of splits is zero
For any pair of odd numbers α,β such that $\alpha+\beta=2^n,\forall n\in\mathbb{N},n\geq 2$, the pair α,β is split recursively until the pair $(1,3)$ is obtained, which is the base point.

Therefore we use the principle of mathematical induction to establish the result for the split for $n\geq 3$.
For $n=3$, there is only one split.
$n=3,2^n=8$ and we have the 4 odd numbers $\{1,3,5,7\}$, we consider those pairs whose sum is 8, that is $(1,7)$ and $(3,5)$.
It is noted that the pair $(3,5)$ is the case of example (2.1) above. Therefore we consider the pair

(1, 7) to obtain

$$\begin{aligned} \varphi(\frac{a+7b}{8}) &= \varphi(\frac{1}{2}(\frac{a+7b}{4})) \\ &= \varphi(\frac{1}{2}([\frac{a+3b}{4}+b])) \\ &\leq \frac{1}{2}\varphi(\frac{a+3b}{4})\varphi(a)+\frac{1}{2}\varphi(b)-\varphi(\frac{1}{8}|b-a|), \end{aligned}$$

using the split and (1.6).

using the base point $\varphi(\frac{a+3b}{4})$, we have

$$\varphi(\frac{1}{4}a+\frac{3}{4}b) \leq \frac{1}{4}\varphi(a)+\frac{3}{4}\varphi(b)-[\frac{1}{2}\varphi(\frac{1}{2}|b-a|)+\varphi(\frac{1}{4}|b-a|)].$$

Hence

$$\varphi(\frac{a+7b}{8}) \leq \frac{1}{2}[\frac{1}{4}\varphi(a)+\frac{3}{4}\varphi(b)-[\frac{1}{2}\varphi(\frac{1}{2}|b-a|)+\varphi(\frac{1}{4}|b-a|)]]+\frac{1}{2}\varphi(b)-\varphi(\frac{1}{8}|b-a|),$$

which simplifies to

$$\varphi(\frac{a+7b}{8}) \leq \frac{1}{8}\varphi(a)+\frac{7}{8}\varphi(b)-[\frac{1}{4}\varphi(\frac{1}{2}|b-a|)+\frac{1}{2}\varphi(\frac{1}{4}|b-a|)+\varphi(\frac{1}{8}|b-a|)].$$

$$=\frac{1}{8}\varphi(a)+\frac{7}{8}\varphi(b)-[\sum_{i=1}^{2}(\frac{1}{2^{3-i}})\varphi(|\frac{b-a}{2^i}|)+\sum_{i=3}^{3}(\frac{1}{2^{3-i}})\varphi(\frac{\epsilon_{(4-i)}(\alpha,\ \beta)}{2^i}|b-a|)].$$

Fix $k \in \mathbb{N}, k > 3$ and suppose (2.2) is true, that is

$\varphi(\frac{\alpha}{2^k}a+\frac{\beta}{2^k}b)$

$\leq \frac{\alpha}{2^k}\varphi(a)+\frac{\beta}{2^k}\varphi(b)-\sum_{i=1}^{2}(\frac{1}{2^{k-i}})\varphi(|\frac{b-a}{2^i}|)-\sum_{i=3}^{k}(\frac{1}{2^{k-i}})\varphi(\frac{\epsilon_{(k+1-i)}(\alpha,\beta)}{2^i}|b-a|)].$

For $k+1 \in \mathbb{N}$, we have $k-1$ splits. Setting $\alpha = (2\delta+1)$ and $\beta = (2\sigma+1)$,

$$\begin{aligned} \varphi[\frac{(2\delta+1)}{2^{k+1}}a+\frac{(2\sigma+1)}{2^{k+1}}b] &= \varphi(\frac{1}{2}[\frac{(2\delta+1)}{2^k}a+\frac{(2\sigma+1)}{2^k}b]) \\ &= \varphi(\frac{1}{2}[\frac{(2\delta+1)a+(2^k-2\delta-1)b}{2^k}+b]), \\ &\leq \frac{1}{2}\varphi(\frac{(2\delta+1)a+(2^k-2\delta-1)b}{2^k})+\frac{1}{2}\varphi(b)-\varphi(\frac{(2\delta+1)}{2^{k+1}}|b-a|), \end{aligned}$$

so

$\varphi[\frac{(2\delta+1)}{2^{k+1}}a+\frac{(2\sigma+1)}{2^{k+1}}b]$

$\leq \frac{1}{2}\varphi(\frac{(2\delta+1)a+(2^k-2\delta-1)b}{2^k})+\frac{1}{2}\varphi(b)-\varphi(\frac{(2\delta+1)}{2^{k+1}}|b-a|)$, using (1.6).

But from the inductive hypothesis, the first term of the immediate inequality becomes:

$\varphi(\frac{(2\delta+1)a+(2^k-2\delta-1)b}{2^k})$

$\leq \frac{(2\delta+1)}{2^k}\varphi(a)+\frac{(2^k-2\delta-1)}{2^k}\varphi(b)-\sum_{i=1}^{2}(\frac{1}{2^{k-i}})\varphi(|\frac{b-a}{2^i}|)-$

$\sum_{i=3}^{k}(\frac{1}{2^{k-i}})\varphi(\frac{\epsilon_{(k+1-i)}(\alpha,\beta)}{2^i}|b-a|).$

Therefore

$\varphi\big(\frac{(2\delta+1)}{2^{k+1}}a + \frac{(2^{k+1}-2\delta-1)}{2^{k+1}}b\big)$

$\leq \frac{(2\delta+1)}{2^{k+1}}\varphi(a) + \frac{(2^{k+1}-2\delta-1)}{2^{k+1}}\varphi(b) - \sum_{i=1}^{2}\big(\frac{1}{2^{k+1-i}}\big)\varphi\big(\big|\frac{b-a}{2^i}\big|\big)-$

$\sum_{i=3}^{k}\big(\frac{1}{2^{k+1-i}}\big)\varphi\big(\frac{\epsilon_{(k+1-i)}(\alpha,\beta)}{2^i}\big|b-a\big|\big) - \varphi\big(\frac{(2\delta+1)}{2^{k+1}}\big|b-a\big|\big)$, which simplifies to

$\varphi\big(\frac{(2\delta+1)}{2^{k+1}}a + \frac{(2^{k+1}-2\delta-1)}{2^{k+1}}b\big)$

$\leq \frac{(2\delta+1)}{2^{k+1}}\varphi(a) + \frac{(2^{k+1}-2\delta-1)}{2^{k+1}}\varphi(b) - \sum_{i=1}^{2}\big(\frac{1}{2^{k+1-i}}\big)\varphi\big(\big|\frac{b-a}{2^i}\big|\big)-$

$\sum_{i=3}^{k+1}\big(\frac{1}{2^{k+1-i}}\big)\varphi\big(\frac{\epsilon_{(k+2-i)}(\alpha,\beta)}{2^i}\big|b-a\big|\big)$,

as required.

Remark 2.1. $\frac{(2\delta+1)}{2^{k+1}}$ is the required minimum before the first split.

Hence inequality (2.2) is true for all $\mathbb{N}$.

Remark 2.2. An inequality similar to that of inequality (2.2) is obtained, when $\alpha+\beta \neq 2^n$.

Proof

Let α, β and $m \in \mathbb{N}$ such that $\alpha+\beta = m \neq 2^n$, where $n \in \mathbb{N}$.
Choose $s \in \mathbb{N}$ such that $2^{s-1} < m < 2^s$ then

$$\begin{aligned}\varphi\big(\frac{\alpha x_1+\beta x_2}{m}\big) &= \varphi\big[\frac{1}{2^s}\big(\alpha x_1+\beta x_2+(2^s-m)\big(\frac{\alpha x_1+\beta x_2}{m}\big)\big)\big] \\ &= \varphi\big[\frac{1}{2}\big(\frac{\alpha x_1+B_1x_2}{2^{s-1}}\big)+\frac{1}{2}\big(\frac{B_2x_2+(2^s-m)\big(\frac{\alpha x_1+\beta x_2}{m}\big)}{2^{s-1}}\big)\big] \\ &\leq \frac{1}{2}\varphi\big(\frac{\alpha x_1+B_1x_2}{2^{s-1}}\big)+\frac{1}{2}\big(\frac{B_2x_2+(2^s-m)\big(\frac{\alpha x_1+\beta x_2}{m}\big)}{2^{s-1}}\big)-\varphi\big(\frac{1}{2}|A-E|\big)\end{aligned}$$

where $B_1, B_2 \in \mathbb{N}$ such that $B_1+B_2=\beta$, $A = \big(\frac{B_2x_2+(2^s-m)\big(\frac{\alpha x_1+\beta x_2}{m}\big)}{2^{s-1}}\big)$ and $E = \big(\frac{\alpha x_1+B_1x_2}{2^{s-1}}\big)$.

But $\varphi\big(\frac{\alpha x_1+B_1x_2}{2^{s-1}}\big)$ simplifies to either inequality (2.1) or (2.2) depending on the values of α and B_1, as shown in Propositions (2.1) and (2.2).

However for the expression $\varphi\big(\frac{B_2x_2+(2^s-m)\big(\frac{\alpha x_1+\beta x_2}{m}\big)}{2^{s-1}}\big)$ we have,

$\varphi\big(\frac{B_2x_2+(2^s-m)\big(\frac{\alpha x_1+\beta x_2}{m}\big)}{2^{s-1}}\big) \leq \frac{B_2}{2^{s-1}}\varphi(x_2)+\frac{(2^s-m)}{2^{s-1}}\varphi\big(\frac{\alpha x_1+\beta x_2}{m}\big)-D,$

where D is the appropriate extra terms.
So
$\varphi\big(\frac{\alpha x_1+\beta x_2}{m}\big) \leq \frac{\alpha}{2^s}\varphi(x_1)+\frac{B_1}{2^s}\varphi(x_2)+\frac{B_2}{2^s}\varphi(x_2)+\frac{(2^s-m)}{2^s}\varphi\big(\frac{\alpha x_1+\beta x_2}{m}\big)-\sum_{i=1}^{s-1-r}\big(\frac{1}{2^{s-i}}\big)\varphi\big(\big|\frac{x_2-x_1}{2^i}\big|\big)-$

$\sum_{i=s-r}^{s-1}\big(\frac{1}{2^{s-i}}\big)\varphi\big(\frac{\epsilon_{(s-i)}}{2^i}\big|x_2-x_1\big|\big)-D,$

$\big(\frac{m}{2^s}\big)\varphi\big(\frac{\alpha x_1+\beta x_2}{m}\big) \leq \frac{\alpha}{2^s}\varphi(x_1)+\frac{\beta}{2^s}\varphi(x_2)-\sum_{i=1}^{s-1-r}\big(\frac{1}{2^{s-i}}\big)\varphi\big(\big|\frac{x_2-x_1}{2^i}\big|\big)-$

$\sum_{i=s-r}^{s-1}\big(\frac{1}{2^{s-i}}\big)\varphi\big(\frac{\epsilon_{(s-i)}}{2^i}\big|x_2-x_1\big|\big)-D.$

Thus

$\varphi\big(\frac{\alpha x_1+\beta x_2}{m}\big) \leq \frac{\alpha}{m}\varphi(x_1) + \frac{\beta}{m}\varphi(x_2) - \frac{2^s}{m}\sum_{i=1}^{s-1-r}\big(\frac{1}{2^{s-i}}\big)\varphi\big(\big|\frac{x_2-x_1}{2^i}\big|\big) -$

$\frac{2^s}{m}\sum_{i=s-r}^{s-1}\big(\frac{1}{2^{s-i}}\big)\varphi\big(\frac{\epsilon(s-i)}{2^i}\big|x_2-x_1\big|\big) - \frac{2^s}{m}D.$

3 Conclusions

A simple calculation points to the fact that the inequality (1.3) which was due to S. Abramovich's definition of superquadratic functions is sharper than the refinement obtained in inequality (2.2) by Smadjor's definition.

However the new inequality obtained is sharper than the original Jensen's inquality but allows the freedom of non-continuity of the superquadratic functions involved.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Banić Senka, et al. Superquadratic functions and refinement of some classical inequalities. J. Korean Math. Soc. 2008;45:513-525.

[2] Abramovich Shashana, et al. Refining jensen's inequality. Bull. Math Soc. Sci. Math. Roumanie. 2004:1-7.

[3] Banić S, Varošanec S. Functional inequalities for superquadratic functions. International Journal of Pure and Applied Mathematics. 2008;43(4):717-729.

[4] Abramovich S. On superquadraticity. Journal of Mathematical Inequalities. 2009;(3):329-339.

[5] Abramovich Shoshana, et al. Inequalities for averages of convex and superquadratic functions. Journal of Inequalities in Pure and Applied Mathe matics. 2004;5(4). Article 91.

[6] Oguntuase JA, Persson LE. Refinement of hardy's inequalities via superquadratic and subquadratic functions. Elsevier Inc. 2007;1305-1312.

[7] Kominek Z, Troczka K. Some remarks on subquadratic functions. Demonstratio Mathematica. 2006;39(4).

[8] Wayne RA, Varberg DE. Convex functions. Academic Press, New York and London. Elsevier Science and Technology Books; 1973.

[9] Niculescu PC, Persson LE. Convex functions and their applications a contemporary approach. Springer, Berlin : Cms Books in Mathematics; 2006.

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