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Series New Exact Solutions to Nonlinear Nizhnik-Novikov-Veselov System

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Abstract

One new solving expression is built for Nizhnik-Novikov-Veselov system in the paper. Through corresponding auxiliary equation arrangement, more than 150 analytical solutions of elementary and Jacobi elliptic functions are obtained so that the NNV system has a wider range of physical meaning. At the same time, the existence and uniqueness of this systematic solution are discussed by fixed point theory of partially ordered space. The expression of the unique solution could be gained if making use of the technique of computer.

Keywords: Nonlinear Nizhnik-Novikov-Veselov System, Solving Expression, Auxiliary Equation, Analytical Solution, Fixed Point Theory of Partially Ordered Space

1. Introduction

(2+1) dimensional nonlinear NizhniK-Novikov-Veselov system is:

$$\begin{cases} u_t + u_{xxx} - 3v_x u - 3vu_x = 0 \\ u_x = v_y \end{cases} \tag{1}$$

One group double-period solutions to (1) are obtained in [1], and the interaction between its various forms is discussed in [2]. One new exact solution is gained in [3], constructed a peak soliton structure with different shape, and the fractal phenomenon of soliton is studied. Reference [4] makes use of (G'/G) expansion method, and three types of new exact solutions (hyperbolic function solution, trigonometric function solution and rational function solution) to (1) are obtained. Through one new solving expression and corresponding auxiliary equation arrangement, more than 150 analytical solutions of elementary and Jacobi elliptic functions could be obtained, which is only given some representative but not all in the paper. So, it makes the NNV system has a wider range of physical meaning. At the same time, the existence and uniqueness of the systematic solution are discussed by fixed point theory of partially ordered space. The expression of unique solution could be gained if making use of the computer technique.

Similar to [4], make traveling-wave transform:

$$u(x,y,t) = u(\xi),\ v(x,y,t) = v(\xi),\ \xi = x + ly - st \tag{2}$$

where: l, s are non-zero constants. Substitute them into (1):

$$\begin{cases} -su' + u''' - 3(uv)' = 0 \\ u' = lv' \end{cases} \tag{3}$$

Integrate the second equation of (3) and get integral constant c_1, we have:

$$u = lv + c_1 \tag{4}$$

Substitute (4) into the first equation of (1) and integrate (get integral constant c_2):

$$lv'' - (3c_1 + sl)v - 3lv^2 - c_2 = 0 \tag{5}$$

2. Analytical Solution

Let:

$$v = a_0 + a_1 f + a_2 f^2 + b_1 f^{-1} + b_2 f^{-2} \tag{6}$$

where: f satisfies

$$f'^2 = e_0 + e_1 f + e_2 f^2 + e_3 f^3 + e_4 f^4 \tag{7}$$

and $a_0, a_1, a_2, b_1, b_2, e_i\ (\mathrm{i} = 0,1,2,3,4)$ are constants to be determined. Substitute (6) and (7) into (5):

$$l\left[2\left(a_2e_0+b_1e_3+3b_2e_4\right)+\frac{1}{2}\left(a_1e_1-3b_1e_3-8b_2e_4\right)\right]$$
$$-\left(3c_1+sl\right)a_0-3l\left(a_0^2+2a_1b_1+2a_2b_2\right)-c_2$$
$$+\left[l\left(3a_2e_1+a_1e_2\right)-3\left(c_1+sl\right)a_1-6l\left(a_0a_1+a_2b_1\right)\right]f$$
$$+\left[l\left(4a_2e_2+\frac{3}{2}a_1e_3\right)-\left(3c_1+sl\right)a_2-3l\left(a_0a_2+a_1^2\right)\right]f^2$$
$$+l\left(5a_2e_3+2a_1e_4-6a_1a_2\right)f^3+3la_2\left(2e_4-a_2\right)f^4$$
$$+\left[l\left(b_1e_2+3b_2e_3\right)-\left(3c_1+sl\right)b_1-6l\left(a_0b_1+a_1b_2\right)\right]f^{-1}$$
$$+\left[l\left(\frac{3}{2}b_1e_1+4b_2e_2\right)-\left(3c_1+sl\right)b_2-3l\left(2a_0b_2+b_1^2\right)\right]f^{-2}$$
$$+l\left(2b_1e_0+5b_2e_1-6b_1b_2\right)f^{-3}+3lb_2\left(2e_0-b_2\right)f^{-4}=0$$

Make the coefficients of the power of f to be zero, and 9 algebraic equations are obtained. Solve them:

$$a_1=e_3 \tag{8.1}$$
$$a_2=2e_4 \tag{8.2}$$
$$b_1=e_1 \tag{8.3}$$
$$b_2=2e_0 \tag{8.4}$$
$$a_0=\frac{l\left(e_1e_2-6e_0e_3\right)-\left(3c_1+sl\right)e_1}{6le_1}\ \text{while}\ e_1\neq 0 \tag{8.5a}$$
$$a_0=\frac{l\left(6e_1e_4-e_2e_3\right)+\left(3c_1+sl\right)e_3}{6le_3}\ \text{while}\ e_3\neq 0 \tag{8.5b}$$
$$c_1=\frac{l}{6e_0}\left(8e_0e_2-\frac{3}{2}e_1^2-12a_0e_0\right)-\frac{sl}{3}\ \text{while}\ e_0\neq 0 \tag{8.6a}$$
$$c_1=\frac{l}{6e_4}\left(8e_2e_4-\frac{3}{2}e_3^2-12a_0e_4\right)-\frac{sl}{3}\ \text{while}\ e_4\neq 0 \tag{8.6b}$$
$$c_2=2l\left(a_2e_0+b_1e_3+3b_2e_4\right)+\frac{l}{2}\left(a_1e_1-3b_1e_3-8b_2e_4\right)$$
$$-\left(3c_1+sl\right)a_0+3l\left(a_0^2+2a_1b_1+2a_2b_2\right) \tag{8.7}$$
$$e_0e_3^2=e_1^2e_4\ \ \text{(from two equations)} \tag{8.8}$$

Solve (7):

1) If $e_0=C^2$, $e_1=2BC$, $e_2=\left(2AC+B^2\right)$, $e_3=2AB$, $e_4=A^2$ then

$$e_0+e_1f+e_2f^2+e_3f^3+e_4f^4=\left(Af^2+Bf+C\right)^2$$

Through separating variables for (7), integrate and arrange:

(a) While $B^2>4AC$, then

$$f=\frac{-B}{2A}+\frac{\sqrt{B^2-4AC}\left[1+e^{\pm\sqrt{B^2-4AC}\left(\xi+\xi_0\right)}\right]}{2A\left[1-e^{\pm\sqrt{B^2-4AC}\left(\xi+\xi_0\right)}\right]} \tag{9.1}$$

(b) While $B^2<4AC$, then

$$f=\frac{\sqrt{4AC-B^2}}{2A}\operatorname{tg}\left[\pm\frac{\sqrt{4AC-B^2}}{2}\left(\xi+\xi_0\right)\right]-\frac{B}{2A} \tag{9.2}$$

(c) While $B^2=4AC$, then

$$f=\frac{1}{2A}\left[\frac{2}{\pm\left(\xi+\xi_0\right)}-B\right] \tag{9.3}$$

2) While $e_i\left(i=0,1,2,3,4\right)$ for different values, equation (7) has different solutions:

(a) $e_0=-1$, $e_1=4r$, $e_2=1-6r^2$, $e_3=-2r\left(1-2r^2\right)$, $e_4=r^2\left(1-r^2\right)$

$$f=\left(\operatorname{sec}h\xi+r\right)^{-1} \tag{10.1}$$

(b) $e_0=-1$, $e_1=-4r$, $e_2=-\left(1-6r^2\right)$, $e_3=2r\left(1-2r^2\right)$, $e_4=-r^2\left(1-r^2\right)$

$$f=\left(\sec\xi+r\right)^{-1}\ \text{or}\ \left(\csc\xi+r\right)^{-1} \tag{10.2}$$

(c) $e_0=1$, $e_1=-4r$, $e_2=-\left(1-4r^2\right)$, $e_3=2r\left(1-2r^2\right)$, $e_4=-r^2\left(1+r^2\right)$

$$f=\left(\operatorname{csc}h\xi+r\right)^{-1} \tag{10.3}$$

(d) $e_0=m^2$, $e_1=-4rm^2$, $e_2=-\left(1+m^2-6r^2m^2\right)$, $e_3=2r\left(1+m^2-2r^2m^2\right)$, $e_4=\left(1-r^2\right)\left(1-r^2m^2\right)$

$$f=\left(\operatorname{sn}\xi+r\right)^{-1} \tag{10.4}$$

(e) $e_0=-m^2$, $e_1=4rm^2$, $e_2=-\left(1-2m^2+6r^2m^2\right)$, $e_3=2r\left(1-m^2+2r^2m^2\right)$, $e_4=\left(1-r^2\right)\left(1-m^2+r^2m^2\right)$

$$f=\left(\operatorname{cn}\xi+r\right)^{-1} \tag{10.5}$$

(f) $e_0=1$, $e_1=4r$, $e_2=4r+m^2$, $e_3=2r\left(2-2r^2-m^2\right)$, $e_4=\left(1-r^2\right)\left(1-r^2-m^2\right)$

$$f=\left(\operatorname{dn}\xi+r\right)^{-1} \tag{10.6}$$

(g) $e_0=e_1=0$, $e_2=q^2$, $e_3=-2rq^2$, $e_4=q^2\left[r^2+\frac{c^2-b^2}{a^2}\right]$

$$f=a\left[b\operatorname{ch}\left(q\xi\right)+c\operatorname{sh}\left(q\xi\right)+ar\right]^{-1} \tag{10.7}$$

(h) $e_0=e_1=0,$ $e_2=-q^2,$ $e_3=2rq^2,$ $e_4=-q^2\left[r^2-\frac{c^2+b^2}{a^2}\right]$

$$f = a\left[b\cos\left(q\xi\right) + c\sin\left(q\xi\right) + ar\right]^{-1} \quad (10.8)$$

Above, "m" in (d), (e) and (f) represents module of Jacobi elliptic function, and the solutions of (g) and (h) can be refer to [5].

3) When $e_1 = e_3 = 0$ and e_0, e_2, e_4 are different values, over 100 kinds of solutions are given by [6] and [7]. For example: while $e_0 = 1$, $e_2 = -2$, $e_4 = 1$, then $f = \mathrm{tg}\xi$; while $e_0 = 1$, $e_2 = -\left(1 + m^2\right)$, $e_4 = m^2$, then $f = \mathrm{sn}\xi$; while $e_0 = m^2/4$, $e_2 = \left(m^2 - 2\right)/2$, $e_4 = m^2/4$, then $f = m\mathrm{sn}\xi/\left(1 + \mathrm{dn}\xi\right)$.

4) When $e_0 = e_4 = 0$ and e_1, e_2, e_3 are different values, 40 kinds of solutions are given by [8]. For example: while $e_1 = 4$, $e_2 = -4\left(1 + m^2\right)$, $e_3 = 4m^2$, then $f = \mathrm{sn}^2\xi$; while $e_1 = -\left(1 - m^2\right)^2$, $e_2 = 2\left(1 + m^2\right)$, $e_3 = -1$, then $f = \left(m\mathrm{cn}\xi \pm \mathrm{dn}\xi\right)^2$.

When coefficients e_0, e_2, e_3, e_4 in auxiliary equation (7) satisfy (8.8), and c_2 takes value from (8.7). Equation (1) has traveling-wave solution according to (4) and (6):

$$\begin{cases} u = lv + c_1 \\ v = a_0 + a_1 f + a_2 f^2 + b_1 f^{-1} + b_2 f^{-2} \end{cases} \quad (11)$$

where: l represents longitudinal wave number, integral constant c_1 takes value from (8.6); according to (8), a_0, a_1, a_2, b_1, b_2 are functions of e_i. Based on the above discussion, there are more than 150 kinds sampling methods, which are not all listed. So, the following are only given a few representative solutions:

1) From this section (b) of (i), while $e_0 = C^2$, $e_1 = 2BC$, $e_2 = \left(2AC + B^2\right)$, $e_3 = 2AB$, $e_4 = A^2$ and $B^2 < 4AC$, then (9.2) is satisfied. Equation (8.8) is:

$$A^2\left(2BC + 2AC + B^2\right)\left(2BC - 2AC - B^2\right) = 0$$

So, Equation (1) has traveling-wave solution according to (8) and (11):

$$\begin{cases} u = lv + c_1 \\ v = \dfrac{l\left(B^2 - 4AC\right) - \left(3c_1 + sl\right)}{6l} \\ \qquad + 2ABf + 2A^2 f^2 + 2BCf^{-1} + 2C^2 f^{-2} \end{cases}$$

where: f takes value from (9.2).

2) From this section (f) of (ii), while $e_0 = 1$, $e_1 = 4r$, $e_2 = 4r + m^2$, $e_3 = 2r\left(2 - 2r^2 - m^2\right)$, $e_4 = \left(1 - r^2\right)\left(1 - r^2 - m^2\right)$, then (10.6) is satisfied and $e_3 = 4r^2 m^2\left(2r^2 - m^2\right) = 0$ is hold based on (8.8). Equation (1) has traveling-wave solution according to (8) and (11):

$$\begin{cases} u = lv + c_1 \\ v = \dfrac{1}{6}\left(6r^2 + 4r - 6 + 4m^2 - s\right) - \dfrac{c_1}{2l} + 2r\left(2 - 2r^2 - m^2\right)f \\ \quad + 2\left(1 - r^2\right)\left(1 - r^2 - m^2\right)f^2 + 4rf^{-1} + 2f^{-2} \end{cases}$$

where: f takes value from (10.6).

3) From this Section 3, while $e_1 = e_3 = 0$, $e_0 = 1$, $e_2 = -\left(1 + m^2\right)$, $e_4 = m^2$, then $f = \mathrm{sn}\xi$, and (8.8) is auto-satisfied. So, the solution to (1) is:

$$\begin{cases} u = lv + c_1 \\ v = 2\left(m^2\mathrm{sn}^2\xi + \mathrm{sn}^{-2}\xi\right) \end{cases}$$

4) From this section (iv), while $e_0 = e_4 = 0$,

$e_1 = -\left(1 - m^2\right)^2$, $e_2 = 2\left(1 + m^2\right)$, $e_3 = -1$, then

$f = \left(m\mathrm{cn}\xi \pm \mathrm{dn}\xi\right)^2$, and (8.8) is auto-satisfied. Because $e_0 = 0$ from (8.6), we know that:

$$e_1 = 0 \Rightarrow m = 1 \Rightarrow \mathrm{cn}\xi \to \sec h\xi, \mathrm{dn}\xi \to \sec h\xi$$
$$\Rightarrow f = 4\sec h^2\xi$$

So, the solution to (1) is:

$$\begin{cases} u = lv + c_1 \\ v = \dfrac{1}{6l}\left(3c_1 + sl - 4l\right) - 4\sec h^2\xi \end{cases}$$

3. Existence and Uniqueness of the Solution

This paper makes use of fixed point theory of partially ordered space [9-11] (author has not found any results better than this from Chinese and foreign literatures for the last decade), and the expression of exact solution to (5) could be obtained combining with the computer technique.

Lemma [9-11] suppose E is real Banach space having normal cone, $u_0 < v_0$, $\left[u_0, v_0\right] \subset E$, then binary operator $A\left(u, v\right)$ is mixed monotone (A increases with the first argument and decreases with the second argument), and satisfies:

1) $u_0 \le A\left(u_0, v_0\right)$, $A\left(v_0, u_0\right) \le v_0$;

2) $\forall u, v$, $u_0 \le u \le v \le v_0$, $\exists \alpha \in \left(0,1\right)$, make $\left\|A\left(v.u\right) - A\left(u, v\right)\right\| \le \alpha\left\|v - u\right\|$.

So the operator A has a unique fixed point $\bar{u}$ ($A\left(\bar{u}, \bar{u}\right) = \bar{u}$). For arbitrary $w_0 \in \left[u_0, v_0\right]$, let $w_{\mathrm{n}} = A\left(w_{\mathrm{n-1}}, w_{\mathrm{n-1}}\right)$ ($\mathrm{n} = 1, 2...$), there is $\bar{u} = \lim_{\mathrm{n}\to\infty} w_{\mathrm{n}}$, and the error estimation is: $\left\|\bar{u} - w_{\mathrm{n}}\right\| \le \alpha^n\left\|v_0 - u_0\right\|$.

Integrate (5), and transfer multiple integral formula into single one:

$$v(\xi)=\frac{1}{l}\int_0^{\xi}(\xi-t)\left[(3c_1+sl)v(t)+3lv^2(t)\right]dt+c_2\xi^2+c_3\xi+c_4 \tag{12.1}$$

where: c_2 is different from that in (5) for one factor. If $l,s>0$ and $c_1<-sl/3$, let

$$A(u,v)=\frac{1}{l}\int_0^{\xi}(\xi-t)\left[(3c_1+sl)v(t)+3lu^2(t)\right]dt+c_2\xi^2+c_3\xi+c_4 \tag{12.2}$$

then binary operator *A* increases with *u* and decreases with *v*, so *A* is mixed monotone. Here it makes $E=C[0,1]$, because $C[0,1]$ is real Banach space having normal cone. Make the equivalent norm as $\|u\|=\max_{0\le t\le 1}e^{-Mt}|u(t)|$ ($M>0$, $u(t)\in C[0,1]$, $[u_0,v_0]=[0,1]$), so

$$A(u_0,v_0)=A(0,1)=\frac{1}{l}\int_0^{\xi}(\xi-t)(3c_1+sl)dt+c_2\xi^2+c_3\xi+c_4$$

$$A(v_0,u_0)=A(1,0)=\frac{1}{l}\int_0^{\xi}3l(\xi-t)dt+c_2\xi^2+c_3\xi+c_4$$

Make integral constant $c_2>-\left(\frac{3c_1+sl}{2l}\right)$, $c_3\ge 0$, $c_4\le-\left(\frac{1}{2}+c_2+c_3\right)$, then

$$u_0=0\le A(u_0,v_0)\le A(v_0,u_0)\le 1=v_0$$

So, operator *A* defined by (12.1) satisfies the condition (i) of Lemma.

$$A(v,u)-A(u,v)$$
$$=\frac{1}{l}\int_0^{\xi}(\xi-t)\left\{3l\left[v(t)+u(t)\right]-(3c_1+sl)\right\}\left[v(t)-u(t)\right]dt$$
$$=\frac{1}{l}\int_0^{\xi}e^{Mt}e^{-Mt}(\xi-t)\left\{3l\left[v(t)+u(t)\right]-(3c_1+sl)\right\}\cdot\left[v(t)-u(t)\right]dt$$

$$\left|A(v,u)-A(u,v)\right|$$
$$\le\frac{1}{l}\int_0^{\xi}e^{Mt}e^{-Mt}(\xi-t)(6l+2lc_2)\left|v(t)-u(t)\right|dt$$
$$\le 2(3+c_2)\|v-u\|\int_0^{\xi}e^{Mt}dt=\frac{2(3+c_2)}{M}\|u-v\|\left(e^{M\xi}-1\right)$$

Make $\alpha=\frac{2(3+c_2)}{M}\in(0,1)$, then

$$\|A(v,u)-A(u,v)\|\le\alpha\|v-u\|$$

Therefore, the binary operator *A* defined by (12.1) satisfies all conditions of Lemma. Based on Lemma, there is one unique continuous function $\overline{u}(\xi)$ for the unique solution to (5) while integral constants c_1, c_2, c_3, c_4 are regular selected. Get $w_0\in[0,1]$ randomly, let

$$w_n(\xi)=\frac{1}{l}\int_0^{\xi}(\xi-t)(3c_1+sl+3l)w_{n-1}(t)\left[1+w_{n-1}(t)\right]dt+c_2\xi^2+c_3\xi+c_4$$

then $\overline{u}(\xi)=\lim_{n\to\infty}w_n(\xi)$, where c_1, c_2, c_3, c_4 are given their values according to the requirements of this section.

4. Conclusions

Accompanied by the corresponding auxiliary equation, this solving expression is built to obtain traveling-wave analytical solution to nonlinear NNV system for the first time. This method for some nonlinear evolution equations is also useful. Some articles solving nonlinear ordinary differential equation by fixed point theory can often be found in various of mathematical journals. In fact, this theory is also very useful for traveling-wave solution to nonlinear evolution equation.

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