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# Conjugate Effects of Viscous Dissipation and Dependent Viscosity on Natural Convection Flow over a Sphere with Joule Heating and Heat Conduction

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***Authors' contributions***

*This work was carried out in collaboration between all authors. Author MMA designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors RI and SG managed the analyses of the study. Author RH managed the literature searches. All authors read and approved the final manuscript.*

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## Abstract

In the present paper, the effect of viscous dissipation and dependent viscosity on free convection flow over a sphere has been investigated. Joule heating and heat conduction over a sphere are considered as well in this investigation. With a goal to attain similarity solutions of the problem being posed, the developed equations are made dimensionless by using suitable transformations. The non-dimensional equations are then transformed into non-linear equations introducing a non- similarity transformation. The resulting non-linear similar equations together with their corresponding boundary conditions based on conduction and convection are solved numerically by using the finite difference method along with Newton's linearization approximation. The numerical results detailing the velocity profiles, temperature profiles, skin friction coefficient and the local heat transfer coefficient are shown both in graph and tabular forms for the different values of the parameters associated with the problem.

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# 1 Introduction

The fundamental problem of free convection flow over a sphere has received considerable attention from researches. The influence of the heat conduction on the free convection flow of the fluid is of paramount importance in various engineering fields. Several numerical and experimental methods have been developed to investigate flow characteristics over the sphere with and without obstacle because these geometries have a wide variety of practical engineering and industrial application, such as in the design of solar collectors, thermal design of building, air conditioning, cooling of electronic, devices, lubrication technologies, chemical processing equipment, drying technologies and so on. With the free convection flow, the phenomenon of the boundary layer flow of an electrically conducting fluid over a sphere in the presence of a Joule-heating term and magnetic field is also very common because of its applications in nuclear engineering in connection with cooling of reactors. Alam et al. [1] investigated the viscous dissipation effects on MHD natural convection flow over a sphere in the presence of heat generation. The effect of viscous dissipation on natural convection flow over a sphere with heat generation is considered by Akter, S., et al. [2]. Miraj, M., et al. [3] discussed the conjugate effects of radiation and viscous dissipation on natural convection flow over a sphere with pressure work. Molla, M. M., et al. [4] have been investigated the effects of temperature dependent thermal conductivity on MHD natural convection flow over an isothermal sphere. The effects of temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plate with heat generation and Joule heating have been examined by Islam, S., et al. [5]. Nasrin, R., et al. [6] performed the combined effects of viscous dissipation and temperature dependent thermal conductivity on magneto hydrodynamic (MHD) free convection flow with conduction and joule heating along a vertical flat plate. Gitima [7] presented the analysis of the effect of variable viscosity and thermal conductivity in micro polar fluid for a porous channel in the presence of magnetic field. Nasrin, R., et al. [8] formulated MHD free convection flow along a vertical flat plate with thermal conductivity and viscosity depending on temperature. Nabil Eldabe T.M., et al. [9] analyzed the effects of temperature dependent viscosity and viscous dissipation on MHD convection flow from an isothermal horizontal circular cylinder in the presence of stress work and heat generation. Safiqul Islam K. M., et al. [10] have been discussed the effects of temperature dependent thermal conductivity on natural convection flow along a vertical flat plate with heat generation. Molla, M. M., et al. [11] analyzed the effect of temperature dependent viscosity on MHD natural convection flow from an isothermal sphere. Alim, M. A., et al. [12] analyzed the heat generation effects on MHD natural convection flow along a vertical wavy surface with variable thermal conductivity. Md. Raihanul Haque et al. [13] established the effects of viscous dissipation on natural convection flow over a sphere with temperature dependent thermal conductivity. Charruaudeau, J., [14] analyzed the influence de gradients de properties physiques en convection force application au cas du tube. Mishra, S. R., et al. [15] have found the numerical solution of a boundary layer MHD flow with viscous dissipation. Acharya, A. K., et al. [16] analyzed free convective fluctuating MHD flow through a porous media past a vertical porous plate with a variable temperature and heat source.

In the present work, we have investigated the viscous dissipation and dependent viscosity effect on the skin friction and the local heat transfer coefficient in the entire region from upstream to downstream of a viscous incompressible and electrically conducting fluid over a sphere in the presence of Joule-heating term. The transformed non similar boundary layer equations governing the flow together with the boundary conditions based on conduction and convection are solved numerically using the implicit finite difference method with Keller box [17] scheme developed by Cebeci and Bradshaw [18] along with Newton's linearization approximation method. We have studied the effect of the Prandtl's number $Pr$, the viscous dissipation parameter $N$, the Joule-heating parameter $J$ and dependent viscosity parameter $\varepsilon$ on the velocity and temperature fields as well as on the skin friction and local heat transfer coefficient. All the investigations for the fluid with a low Prandtl's number being appropriate for the liquid metals are carried out.

## 2 Formulation of the Problem

We consider a steady two-dimensional natural convection boundary layer flow of an electrically conducting and viscous incompressible fluid over a sphere of radius $a$ , $H_0$ is the magnetic field strength and $\sigma$ is the electrical conductivity. The surface temperature of the sphere is assumed as $T_w$ and $T_\infty$ being the ambient temperature of the fluid. When $T_w > T_\infty$ an upward flow is established along the surface due to free convection and the flow is downward for $T_w < T_\infty$. The mathematical model for the assumed physical problem is prescribed by the following conservation equation of mass, momentum and energy.

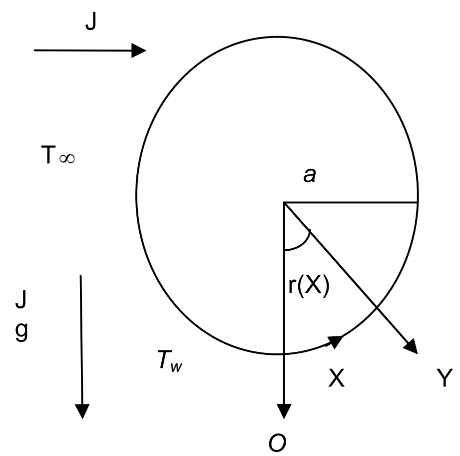


**Fig. 1. Physical model and coordinate system**

Under these considerations the governing equations are

$$\frac{\partial}{\partial X}(rU)+\frac{\partial}{\partial Y}(rV)=0 \tag{1}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=\frac{1}{\rho}\frac{\partial}{\partial Y}\left(\mu\frac{\partial U}{\partial Y}\right)+g\beta\left(T-T_\infty\right)\sin\left(\frac{X}{a}\right) \tag{2}$$

$$U\frac{\partial T}{\partial X}+V\frac{\partial T}{\partial Y}=\frac{K}{\rho C_p}\frac{\partial^2 T}{\partial Y^2}+\frac{\nu}{\rho C_p}\left(\frac{\partial U}{\partial Y}\right)^2+\frac{\sigma H_0^2}{\rho C_p}U^2 \tag{3}$$

The boundary conditions for the governing equations are

$$\left.\begin{array}{l} U=V=0, \quad T=T_w \quad on \quad Y=0 \\ U\to 0, T\to T_\infty \quad at \quad Y\to\infty \end{array}\right\} \tag{4}$$

$$r(X) = a \sin\left(\frac{X}{a}\right) \tag{5}$$

Where $r$ is the radial distance from the symmetrical axis to the surface of the sphere. Here, we will consider $\mu = \dfrac{\mu_\infty}{1+\alpha(T-T_\infty)}$ is the dependent viscosity where $\alpha = \dfrac{1}{\mu_f}\left(\dfrac{\delta\mu}{\delta T}\right)_f$.

# 3 Transform of the Governing Equations

The above equations are non-dimensional down usual manner by the following substitutions:

$$\xi = \frac{X}{a},\ \eta = \mathrm{Gr}^{1/4}\ \frac{Y}{a},\ u = \frac{U}{u_0} = \frac{a}{\nu}\mathrm{Gr}^{-1/2}\ U,\ v = \frac{a}{\nu}\mathrm{Gr}^{-1/4}V,$$
$$\theta = \frac{T-T_\infty}{T_w-T_\infty},\quad \theta_w = \frac{T_w}{T_\infty} \tag{6}$$

Where, $u_0 = \dfrac{\upsilon}{a}Gr^{1/2}$ is the characteristic velocity of the fluids. Here we will consider

$$\beta = -\frac{1}{\rho}\left(\frac{\delta\rho}{\delta T_f}\right)_p,\ Gr = \frac{g\beta a^3}{\upsilon^2}\left(T_w - T_\infty\right),$$
$$\varepsilon = \frac{1}{\mu_f}\left(\frac{\delta\mu}{\delta T}\right)_f\left(T-T_\infty\right),\ \frac{\mu}{\mu_\infty} = \frac{1}{1+\varepsilon\theta}$$

Using the above transformations into equations (1) to (3), we have

$$\therefore \frac{\delta}{\delta\xi}(ru) + \frac{\delta}{\delta\eta}(rv) = 0 \tag{7}$$

$$u\frac{\partial u}{\partial\xi} + v\frac{\partial u}{\partial\eta} = \frac{-\varepsilon}{(1+\varepsilon\theta)^2}\frac{\partial u}{\partial\eta}\frac{\partial\theta}{\partial\eta} + \frac{1}{1+\varepsilon\theta}\frac{\partial^2 u}{\partial\eta^2} + \theta\sin\xi \tag{8}$$

$$u\frac{\partial\theta}{\partial\xi} + \upsilon\frac{\partial\theta}{\partial\eta} = \frac{1}{Pr}\frac{\partial^2\theta}{\partial\eta^2} + \frac{Gr}{a^2C_P\left(T_w-T_\infty\right)}\left(\frac{\partial u}{\partial\eta}\right)^2 + \frac{\sigma H_0^2 ag\beta\left(T_w-T_\infty\right)}{\rho C_p}u^2$$

$$u\frac{\partial\theta}{\partial\xi} + \upsilon\frac{\partial\theta}{\partial\eta} = \frac{1}{Pr}\frac{\partial^2\theta}{\partial\eta^2} + N\left(\frac{\partial u}{\partial\eta}\right)^2 + J\left(u^2\right) \tag{9}$$

The boundary conditions associated with (8) to (9) becomes

$$\left.\begin{aligned} &u = v = 0, \quad \theta = 1 \quad at\ \xi = 0,\ for\ any\ \eta \\ &u = v = 0, \quad \theta = 1 \quad at\ \eta = 0, \xi > 0 \\ &u \to 0, \quad \theta \to 0\ \ as\ \eta \to \infty,\ \ \xi > 0 \end{aligned}\right\} \tag{10}$$

Here, $Gr = g\beta(T_w - T_\infty)a^3/\nu^2$ is the Grashof number and $\theta$ is the non-dimensional temperature function, $\Pr = \frac{\mu C_p}{k_\infty}$ is the Prandtl's number, and $\varepsilon = \frac{1}{\mu_f}\left(\frac{\delta\mu}{\delta T}\right)_f (T - T_\infty)$ is the dependent viscosity parameter. Also $N = \frac{Gr}{a^2 C_p (T_w - T_\infty)}$ is the viscous dissipation parameter, $J = \frac{\sigma H_o^2 ag\beta(T_w - T_\infty)}{\rho C_p}$ is the Joule heating parameter. To solve equations (8) and (9) are subjected to the boundary conditions (10), where we assume the following variables $u = \frac{1}{r}\frac{\partial\psi}{\partial\eta}$ and $v = -\frac{1}{\mathrm{r}}\frac{\partial\psi}{\partial\xi}$ where $\psi = \xi\ r(\xi) f(\xi,\eta)$ and $\psi(\xi, \eta)$ is a non-dimensional stream function which is related to the velocity components in the usual way as $u = \frac{1}{r}\frac{\partial\psi}{\partial\eta}$ and $v = -\frac{1}{\mathrm{r}}\frac{\partial\psi}{\partial\xi}$

The momentum and energy equations (8) to (9) reduce to

$$\frac{1}{1+\varepsilon\theta}\frac{\delta^3 f}{\delta\eta^3} + \left(1 + \frac{\xi}{\sin\xi}\cos\xi\right) f\frac{\delta^2 f}{\delta\eta^2} - \left(\frac{\delta f}{\delta\eta}\right)^2 - \frac{\varepsilon}{(1+\varepsilon\theta)^2}\frac{\delta\theta}{\delta\eta}\frac{\delta^2 f}{\delta\eta^2} \tag{11}$$

$$+\frac{\theta\sin\xi}{\xi} = \xi\left(\frac{\delta f}{\delta\eta}\frac{\delta^2 f}{\delta\eta\delta\xi} - \frac{\delta f}{\delta\xi}\frac{\delta^2 f}{\delta\eta^2}\right)$$

$$\frac{1}{\Pr}\frac{\partial^2\theta}{\partial\eta^2} + \left(1 + \frac{\xi}{\sin\xi}\cos\xi\right) f\frac{\partial\theta}{\partial\eta} + N\xi^2\left(\frac{\partial^2 f}{\partial\eta^2}\right)^2 \tag{12}$$

$$J\left(\frac{\partial f}{\partial\eta}\right)^2 = \xi\left(\frac{\partial f}{\partial\eta}\frac{\partial\theta}{\partial\xi} - \frac{\partial f}{\partial\xi}\frac{\partial\theta}{\partial\eta}\right)$$

The corresponding boundary conditions are

$$\left.\begin{aligned} &f(\xi,\eta) = f'(\xi,\eta) = 0, \quad \theta = 1\ at\ \eta = 0 \\ &f(\xi,\eta) = f'(\xi,\eta) = 0, \quad \theta = 1\ at\ \eta = 0, \xi > 0 \\ &f'(\xi,\eta) \to 0, \theta \to 0\ as\ \eta \to \infty, \xi > 0 \end{aligned}\right\} \tag{13}$$

In practical application, the physical quantities of principal interest are the heat transfer and the skin- friction coefficient, which can be written in non- dimensional form as

$$Nu_{\xi} = \frac{aGr^{-1/4}}{k\left(T_w - T_{\infty}\right)} q_w \text{ and } Cf_{\xi} = \frac{Gr^{-3/4} a^2}{\mu \nu} \tau_w \tag{14}$$

Where $q_w = -k_f \left(\frac{\partial T}{\partial Y}\right)_{Y=0}$ and $\tau_w = \mu \left(\frac{\partial U}{\partial Y}\right)_{Y=0}$, $k_f$ being the thermal conductivity of the fluid. Using the new variables (6), we have the simplified form of the heat transfer and the skin- friction coefficient as

$$Nu_{\xi} = -\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=0} \text{ and } Cf_{\xi} = \xi \left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\eta=0} \tag{15}$$

# 4 Method of Solution

This paper deals with the natural convection flow on variation viscosity and viscous dissipation of viscous incompressible fluid over a heated sphere with Joule heating and magneto hydrodynamics being investigated using the very efficient implicit finite difference method known as the Keller box scheme developed by Keller [17], which has been well documented by Cebeci and Bradshaw [18].

To apply the aforementioned method, equations (11) and (12) with their boundary conditions (13) are first converted into the following system of first order equations. For this purpose we introduce new dependent variables $u\left(\xi,\eta\right)$, $v\left(\xi,\eta\right)$, $p\left(\xi,\eta\right)$ and $g\left(\xi,\eta\right)$ so that the transformed momentum and energy equations can be written as

$$f' = u \tag{16}$$

$$u' = v \tag{17}$$

$$g' = p \tag{18}$$

$$P_1 v' + P_2 f\, u' - u^2 - P_3 g u' + P_4 g = \xi\left(u\frac{\partial u}{\partial \xi} - v\frac{\partial f}{\partial \xi}\right) \tag{19}$$

$$\frac{1}{\Pr} p' + P_1 f\ p + P_5 \left(u'\right)^2 + P_6 u^2 = \xi\left(u\frac{\partial g}{\partial \xi} - p\frac{\partial f}{\partial \xi}\right) \tag{20}$$

where $x = \xi, \theta = g$ and $P_1 = \frac{1}{1+\varepsilon\theta}$, $P_2 = 1 + \frac{\xi\cos\xi}{\sin\xi}$, $P_3 = \frac{\varepsilon}{\left(1+\varepsilon\theta\right)^2}$,

$P_4 = \frac{\sin\xi}{\xi}$, $P_5 = N\xi^2$, $P_6 = J$

and the boundary conditions (14) are

$$\begin{aligned} &f(\xi,0)=0, \quad u(\xi,0)=v(\xi,0)=0, g(\xi,0)=1, \xi \rangle 0 \\ &u(0,\eta)=v(0,\eta)=0, \quad g(0,\eta)=1 \\ &u \to 0, v \to 0 \; as \; \eta \to \infty, \xi \rangle 0 \end{aligned} \tag{21}$$

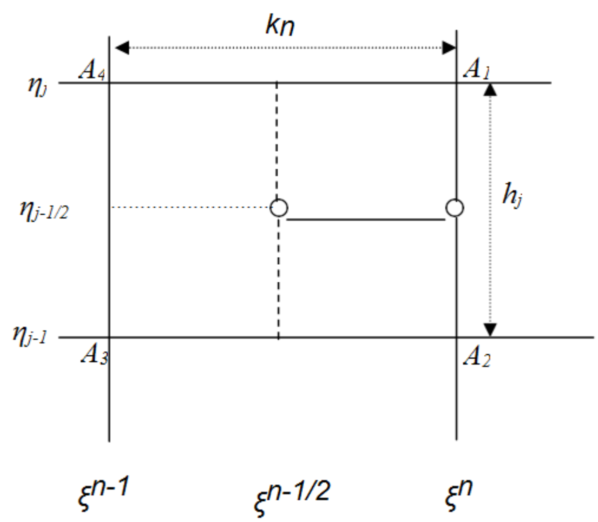


**Fig. 2. Net rectangle of difference approximations for the Box scheme**

Now, we consider the net rectangle on the $(\xi,\eta)$ plane as shown in the Fig. 2 and denote the net points by

$$\xi^0=0, \xi^n=\xi^{n-1}+k_n, n=1,2,3,—,N \; and \; \eta_0=0, \eta_j=\eta_{j-1}+h_j, j=1,2,3,—,J \tag{22}$$

Here, $n$ and $j$ are just the sequence of numbers on the $(\xi,\eta)$ plane, $k_n$ and $h_j$ are the variable mesh widths. Approximate the quantities $f$, $u$, $v$ and $p$ , at the points $(\xi^n,\eta_j)$ of the net by $f_j^n$, $u_j^n$, $v_j^n$, $p_j^n$ which is called net function. It is also employed so that the notation $P_j^n$ for the quantities midways between net points as shown in Fig. 2 and for any net functions as

$$\xi^{n-\frac{1}{2}}=\frac{1}{2}\left(\xi^n+\xi^{n-1}\right) \tag{23}$$

$$\eta_{j-1/2}=1/2\left(\eta_j+\eta_{j-1/2}\right) \tag{24}$$

$$g_j^{n-1/2}=\frac{1}{2}\left(g_j^n+g_j^{n-1}\right) \tag{25}$$

$$g_{j-1/2}^n=\frac{1}{2}\left(g_j^n+g_{j-1}^n\right) \tag{26}$$

The finite difference approximation according to box method to the three first order ordinary differential equations (18)-(20) is written for the mid - point $\left(\xi^n, \eta_{j-1/2}\right)$ of the segment $A_1A_2$ as shown in Fig. 2.

$$\frac{f_j^n - f_{j-1}^n}{h_j} = u_{j-1/2}^n = \frac{u_{j-1}^n + u_j^n}{2} \tag{27}$$

$$\frac{u_j^n - u_{j-1}^n}{h_j} = v_{j-1/2}^n = \frac{v_j^n + v_{j-1}^n}{2} \tag{28}$$

$$\frac{g_j^n - g_{j-1}^n}{h_j} = p_{j-1/2}^n = \frac{p_j^n + p_{j-1}^n}{2} \tag{29}$$

The finite difference approximation to the first order differential equation (20) and (21) is written for the mid - point $\left(\xi^{n-1/2}, \eta_{j-1/2}\right)$ of the rectangle $A_1A_2A_3A_4$. This procedure yields

$$\begin{aligned}&\frac{1}{2}\left(P_1\frac{v_j^n - v_{j-1}^n}{h_j}\right) + \frac{1}{2}\left(P_1\frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + \left(P_2 f\, v\right)_{j-1/2}^{n-1/2} - \left(u^2\right)_{j-1/2}^{n-1/2} - \left(P_3 g v\right)_{j-1/2}^{n-1/2} + \left(P_4 g\right)_{j-1/2}^{n-1/2}\\ &= \xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\frac{u_{j-1/2}^n - u_{j-1/2}^{n-1}}{k_n} - v_{j-1/2}^{n-1/2}\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right)\end{aligned} \tag{30}$$

$$\begin{aligned}&\frac{1}{2\Pr}\left(\frac{p_j^n - p_{j-1}^n}{h_j}\right) + \frac{1}{2\Pr}\left(\frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) + \left(P_1 f\, p\right)_{j-1/2}^{n-1/2}\\ &+ \left(P_5\, v^2\right)_{j-1/2}^{n-1/2} + \left(P_6\, u^2\right)_{j-1/2}^{n-1/2} = \xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\frac{g_{j-1/2}^n - g_{j-1/2}^{n-1}}{k_n} - p_{j-1/2}^{n-1/2}\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right)\end{aligned} \tag{31}$$

The above equations are to be linearized using Newton's Quasi-linearization method. Then linear algebraic equations can be written in block matrix which forms a coefficient matrix. The whole procedure, that includes, reduction to first order followed by central difference approximations, Newton's Quasi-linearization method and the block Thomas algorithm, is well known as the Keller-box method.

# 5 Results and Discussion

Here, we have investigated the effect of viscous dissipation and dependent viscosity on free convection flow over a sphere in the presence of Joule-heating and heat conduction.

Solutions are obtained for the fluids having Prandtl's number *Pr* = 0.72, 3.00, 5.00, 7.00, and a wide range of the values of the viscous dissipation parameter *N* = 0.10, 0.40, 0.70, 0.90 and the Joule-heating parameter *J* = 1.60, 1.10, 0.70, 0.30. If we know the values of the functions $f(\xi,\eta)$, $\theta(\xi,\eta)$ and their derivatives for the different values of the Prandtl's number *Pr* and the Joule-heating parameter *J*, we may calculate the numerical values of the local heat transfer coefficient $\theta'(x,0)$ and the velocity gradient $f''(\xi,\eta)$ on the surface that are important from the physical point of view.

Fig. 3(a) and Fig. 3(b) deal with the effect of the viscous dissipation parameter $N$ (= 0.10, 0.40, 0.60, 0.80) for different values of the controlling parameters $Pr$ = 0.72, $J$ = 0.80 and ε = 2.00 on the velocity profiles $f'(\xi,\eta)$ and the temperature profiles $\theta(\xi,\eta)$. From Fig. 3(a), it is revealed that the velocity profile $f'(\xi,\eta)$ increases very slightly with the increase in the viscous dissipation parameter $N$ which indicates that viscous dissipation increases the fluid motion slowly. From Fig. 3(b), it is shown that there is a small increase in temperature profiles $\theta(\xi,\eta)$ for the increasing values of $N$ with other controlling parameter.

From Fig. 4(a), it is observed that an increase in the Joule heating parameter is associated with a considerable increase in velocity profiles, But near the surface of the plate the velocity increases and becomes maximum and then decreases and finally approaches to zero. Fig. 4(b) shows the distribution of the temperature profiles $\theta(\xi,\eta)$ against $\eta$ for the same values of the Joule heating parameter $J$ and each of which attains maximum at the surface.

Fig. 5(a) and 5(b) deal with the effect of the dependent viscosity parameter ε (= 0.30, 0.70, 1.10, 1.60) with other controlling parameters $Pr$ = 0.72, $N$ = 0.50 and $J$ = 0.40 on the velocity profile $f'(\xi,\eta)$ and the temperature profile $\theta(\xi,\eta)$. From Fig. 5(a), it is revealed that the velocity profile $f'(\xi,\eta)$ increases very slightly with the increase in the dependent viscosity parameter $\varepsilon$ which indicates that dependent viscosity parameter increases the fluid motion slowly. From Fig. 5(b), it is shown that the temperature profiles $\theta(\xi,\eta)$ increase for the increasing values of dependent viscosity parameter $\varepsilon$.

Fig. 6(a) depicts the velocity profile for the different values of the Prandtl's number, $Pr$ (= 0.72, 3.00, 5.00, 7.00) while the others controlling parameters $N$ = 0.50, $J$ = 0.60 and $\varepsilon$ = 1.00. The corresponding distribution of the temperature profile $\theta(\xi,\eta)$ in the fluids is shown in Fig. 6(b). From Fig. 6(a), it is seen that if the Prandtl's number increases, the velocity of the fluid decreases. On the other hand, from Fig. 6(b) it is observed that the temperature profile decreases within the boundary layer due to the increase in the Prandtl's number $Pr$.

Numerical values of the velocity gradient $f''(\xi,0)$ and the local heat transfer coefficient $\theta'(\xi,0)$ are depicted graphically in Fig. 7(a) and 7(b) respectively against the axial distance $\xi$ for different values of the viscous dissipation parameter $N$ (= 0.10, 0.40, 0.60, 0.80) for the fluid having Prandtl number $Pr$ = 0.72, $J$ = 0.80 and $\varepsilon$ = 2.00. It is seen from Fig. 7(a) that the skin-friction $f''(\xi,0)$ increases when the viscous dissipation parameter $N$ increases. It is also observed in Fig. 7(b) that the local heat transfer coefficient $\theta'(\xi,0)$ deceases as viscous dissipation parameter $N$ increases.

The effect of Joule heating parameter $J$ (= 0.10, 0.30, 0.60, 0.90) on the skin-friction $f''(\xi,0)$ and the surface temperature distribution $\theta(\xi,0)$ against $\xi$ for $Pr$ = 0.72, $N$ = 0.60 and ε = 1.00 is shown in Fig. 8(a) - 8(b). It is found that both values of the skin-friction $f''(\xi,0)$ and the local heat transfer coefficient $\theta'(\xi,0)$ increase for the increasing values of Joule heating parameter $J$. Here it has been observed that the values of the skin-friction $f''(\xi,0)$ increases by 76.734% and the local heat transfer coefficient $\theta'(\xi,0)$ increases by 83.264% while $J$ increased from 0.10 to 0.90.

From Fig. 9(a), it is observed that increase in the value of the dependent viscosity parameter $\varepsilon$ leads to an increase in the value of the shear stress coefficient $f''(\xi,0)$ which is usually expected. Again from Fig. 9(b) it is illustrated that the increase in the dependent viscosity parameter *J* leads to an increase in the local heat transfer coefficient $\theta'(\xi,0)$.

From Fig. 10(a), it is be observed that an increase in the value of the Prandtl's number *Pr* (= 0.72, 3.00, 5.00, 7.00) leads to a decrease in the value of shear stress $f''(\xi,0)$. Similar results hold good in local heat transfer coefficient $\theta'(\xi,0)$ as shown in Fig. 10(b) for the same values of Prandtl's number *Pr* when $J$ = 0.60, $N$ = 0.50 and $\varepsilon$ = **1**.00.

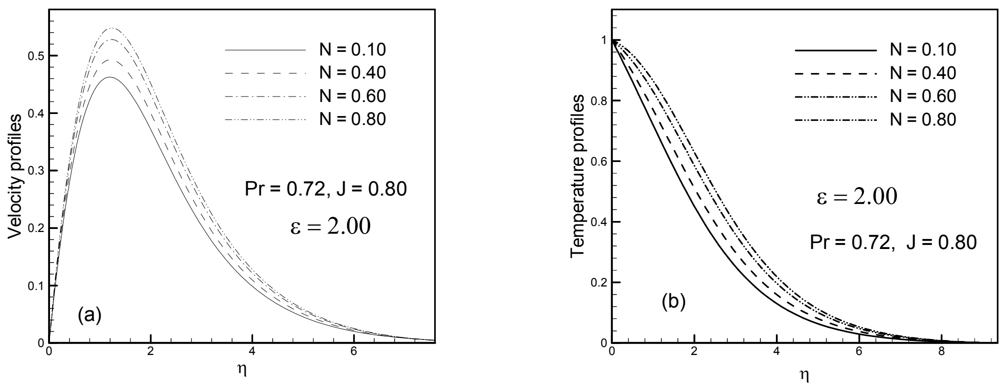


**Fig. 3(a) and 3(b). Variation of dimensionless velocity profiles $f'(\eta,\xi)$ and temperature profiles $\theta(\eta,\xi)$ against dimensionless distance $\eta$ for different values of viscous dissipation parameter *N* with *Pr* = 0.72, $\varepsilon$ = 2.00 and *J* = 0.80**

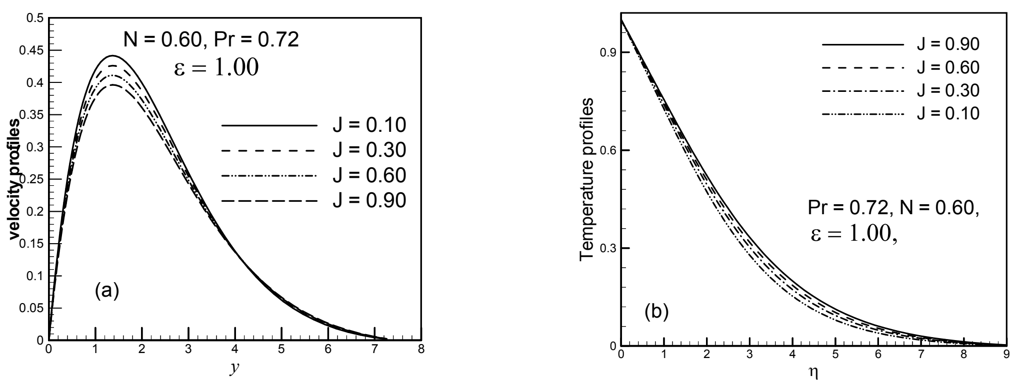


**Fig. 4(a) and 4(b). Variation of dimensionless velocity profiles $f'(\eta,\xi)$ and temperature profiles $\theta(\eta,\xi)$ against dimensionless distance $\eta$ for different values of Joule heating parameter *J* with *Pr* = 0.72, $\varepsilon$ = 1.00 and *N*= 0.60**

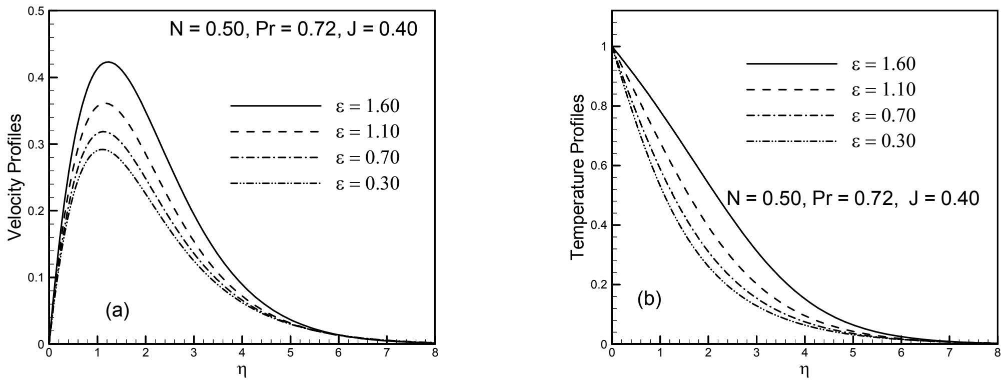


**Fig. 5(a) and 5(b). Variation of dimensionless velocity profiles $f'(\eta,\xi)$ and temperature profiles $\theta(\eta,\xi)$ against dimensionless distance $\eta$ for different values of dependent viscosity parameter $\varepsilon$ with *Pr* = 0.72, *N* = 0.60 and *J* = 0.40**

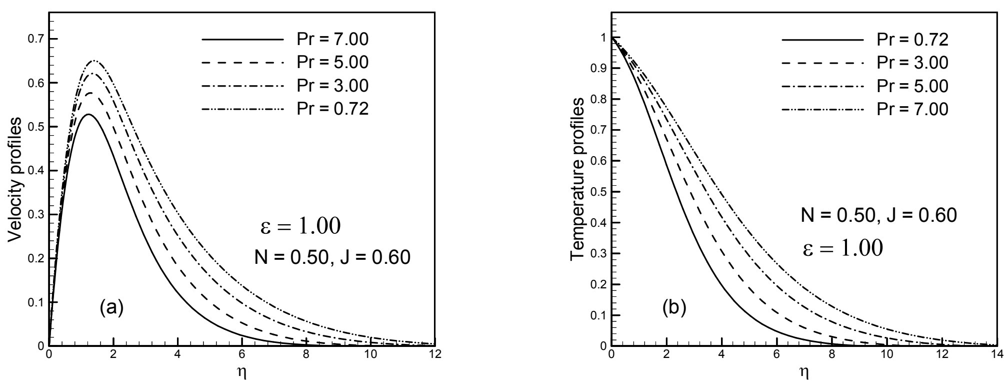


**Fig. 6(a) and 6(b). Variation of dimensionless velocity profiles $f'(\eta,\xi)$ and temperature profiles $\theta(\eta,\xi)$ against dimensionless distance $\eta$ for different values of Prandtl's number *Pr* with *N* = 0.50, $\varepsilon$ = 1.00 and *J* = 0.60**

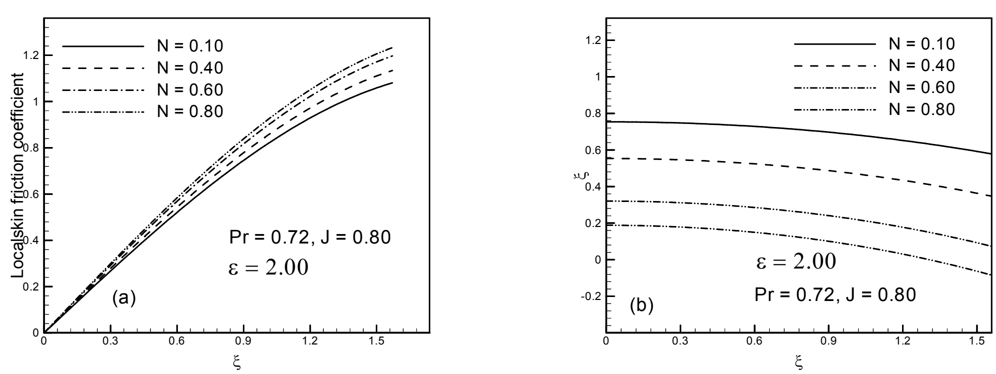


**Fig. 7(a) and 7(b). Variation of dimensionless skin friction coefficient $f''(\xi,0)$ and local Nusselt number, $\theta'(\xi,0)$ against dimensionless distance $\xi$ for different values of viscous dissipation parameter *N* with *Pr* = 0.72, *J* = 0.80 and $\varepsilon$ = 2.00**

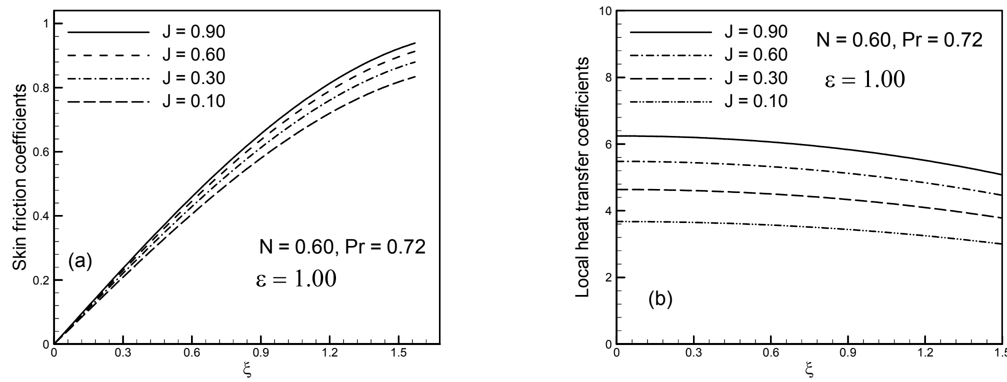


**Fig. 8(a) and 8(b). Variation of dimensionless skin friction coefficient $f''(\xi,0)$ and local heat transfer coefficient $\theta'(\xi,0)$ against dimensionless distance $\xi$ for different values of Joule heating parameter *J* with *Pr* = 0.72, $\varepsilon$ = 1.00 and *N* = 0.60**

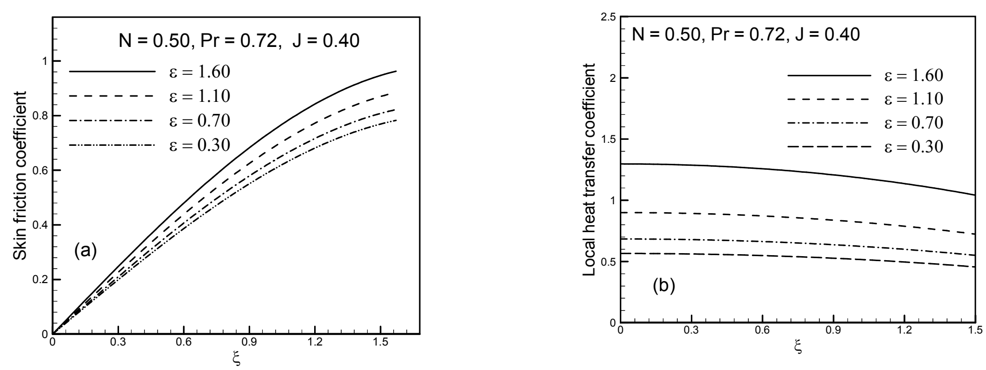


**Fig. 9(a) and 9(b). Variation of dimensionless skin friction coefficient $f''(\xi,0)$ and local heat transfer coefficient $\theta'(\xi,0)$ against dimensionless distance $\xi$ for different values of dependent viscosity parameter $\varepsilon$ with *Pr* = 0.72, *J* = 0.40 and *N* = 0.50**

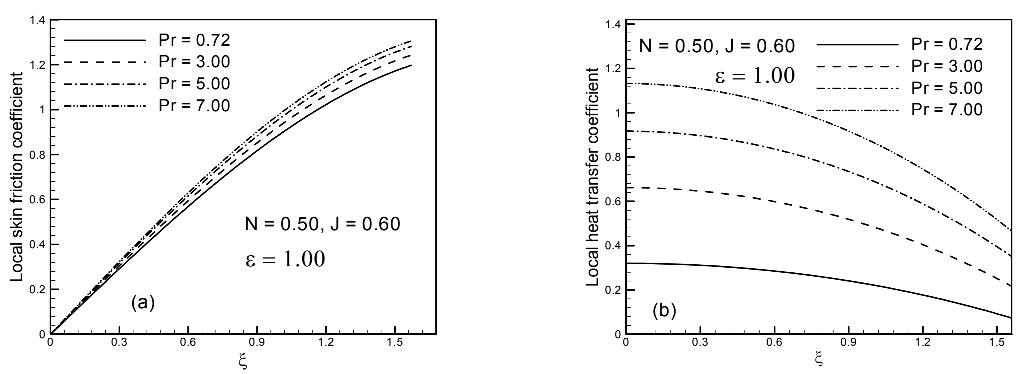


**Fig. 10(a) and 10(b). Variation of dimensionless skin friction coefficient $f''(\xi,0)$ and local heat transfer coefficient $\theta'(\xi,0)$ against dimensionless distance $\xi$ for different values of Prandtl's number *Pr* with *N* = 0.50, *J* = 0.60 and $\varepsilon$ = 1.00**

In Table 1 given the tabular values of the local skin friction coefficient $f''(\xi,0)$ and local Nusselt number $\theta'(\xi,0)$ for different values of viscous dissipation parameter $N$ while $Pr$ =0.72 , Joule heating parameter $J$ = 0.90 and $\varepsilon$ = 2.00. Here it is found that the values of local skin friction coefficient $f''(\xi,0)$ increase at different position of $\xi$ for viscous dissipation parameter $N$ = 0.10, 0.40, 0.60, 0.80. The rate of local skin friction coefficient $f''(\xi,0)$ increases by 24.76% as the viscous dissipation parameter $N$ changes from 0.10 to 0.80 and $\xi$=0.87266. Furthermore, it is seen that the numerical values of the local rate of heat transfer $\theta'(\xi,0)$ decrease for the increasing values of viscous dissipation parameter $N$.The rate of increase the local rate of heat transfer is 49.86% at position $\xi$=0.34907 as the viscous dissipation parameter $N$ changes from 0.10 to 0.80.Numerical values of local heat transfer $\theta'(\xi,0)$ are calculated from equation (15) for the surface of the sphere from lower stagnation point to upper stagnation point.

**Table 1. Skin friction coefficient and rate of heat transfer against $\xi$ for different values of viscous dissipation parameter N with other controlling parameters Pr = 0.72, $\varepsilon$ = 2.00, J = 0.90**

| | ***N* = 0.10** | | ***N* = 0.40** | | ***N* = 0.60** | | ***N* = 0.80** | |
|---|---|---|---|---|---|---|---|---|
| $\xi$ | $f''(\xi,0)$ | $\theta'(\xi,0)$ | $f''(\xi,0)$ | $\theta'(\xi,0)$ | $f''(\xi,0)$ | $\theta'(\xi,0)$ | $f''(\xi,0)$ | $\theta'(\xi,0)$ |
| 0.00000 | 0.00000 | 0.84411 | 0.00000 | 1.12538 | 0.00000 | 1.24695 | 0.00000 | 1.35958 |
| 0.17453 | 0.16133 | 0.62493 | 0.17012 | 0.85257 | 0.17318 | 0.95101 | 0.17573 | 1.04203 |
| 0.34907 | 0.31963 | 0.61047 | 0.33720 | 0.83443 | 0.34325 | 0.93137 | 0.34829 | 1.02100 |
| 0.52360 | 0.47246 | 0.59545 | 0.49809 | 0.81527 | 0.50700 | 0.91047 | 0.51439 | 0.99829 |
| 0.69813 | 0.61656 | 0.57720 | 0.64982 | 0.79130 | 0.66133 | 0.88403 | 0.67091 | 0.96969 |
| 0.87266 | 0.74952 | 0.55438 | 0.78949 | 0.76152 | 0.80323 | 0.85126 | 0.81486 | 0.93401 |
| 1.04720 | 0.86886 | 0.52677 | 0.91431 | 0.72533 | 0.93016 | 0.81124 | 0.94334 | 0.89061 |
| 1.22173 | 0.97169 | 0.49398 | 1.02172 | 0.68229 | 1.03911 | 0.76374 | 1.05359 | 0.83897 |
| 1.39626 | 1.05588 | 0.45560 | 1.10912 | 0.63190 | 1.12758 | 0.70816 | 1.14300 | 0.77854 |
| 1.57080 | 1.11937 | 0.41125 | 1.17409 | 0.57368 | 1.19312 | 0.64392 | 1.20900 | 0.70870 |

**Table 2. Comparisons of the present numerical results of $Nu_\xi$ for the Prandtl numbers *Pr* = 0.7, 7.0 without effect of the viscous dissipation parameter, joule heating parameter and dependent viscosity parameter with those obtained by Molla et al. [19] and Nazar et al. [20]**

| **$\xi$ in degree** | ***Pr* = 0.70** | | | ***Pr* = 7.00** | | |
|---|---|---|---|---|---|---|
| | **Naza et al. [20]** | **Molla et al. [19]** | **Present** | **Naza et al. [20]** | **Molla et al. [19]** | **Present** |
| 0 | 0.4576 | 0.4576 | 0.4492 | 0.9595 | 0.9582 | 0.9527 |
| 10 | 0.4565 | 0.4564 | 0.4485 | 0.9572 | 0.9558 | 0.9487 |
| 20 | 0.4533 | 0.4532 | 0.4467 | 0.9506 | 0.9492 | 0.9410 |
| 30 | 0.4480 | 0.4479 | 0.4401 | 0.9397 | 0.9383 | 0.9279 |
| 40 | 0.4405 | 0.4404 | 0.4340 | 0.9239 | 0.9231 | 0.9140 |
| 50 | 0.4308 | 0.4307 | 0.4259 | 0.9045 | 0.9034 | 0.8945 |
| 60 | 0 .4189 | 0.4188 | 0.4106 | 0.8801 | 0.8791 | 0.8611 |
| 70 | 0.4046 | 0.4045 | 0.3982 | 0.8510 | 0.8501 | 0.8395 |
| 80 | 0.3879 | 0.3877 | 0.3769 | 0.8168 | 0.8161 | 0.8133 |
| 90 | 0.3684 | 0.3683 | 0.3607 | 0.7774 | 0.7768 | 0.7707 |

The comparisons of the local heat transfer coefficient between the present work and the work of Nazar et al. [20] and Molla et al. [19] are presented in Table 2 respectively. We observe that the comparison without the effect of viscous dissipation parameter, dependent viscosity parameter and Joule heating parameter in the present problem is similar to the previous work.

## 6 Conclusions

From the present investigation, the following conclusions may be drawn:

- Increase in the values of viscous dissipation parameter $N$ leads to an increase in the velocity profile, the temperature profile, the local skin friction coefficient $f''(\xi,0)$ but the local rate of heat transfer $\theta'(\xi,0)$ decreases with the increase in viscous dissipation parameter $N$ for $J$ =0.80, $\varepsilon$ = 2.00 and $Pr$ = 0.72.
- The velocity profiles, the temperature profiles, the local skin friction coefficient $f''(\xi,0)$ and also the local heat transfer coefficient $\theta'(\xi,0)$ increase significantly when the values of dependent viscosity parameter $\varepsilon$ increase.
- Significant effects of Joule heating parameter $J$ on velocity and temperature profiles as well as on local skin friction coefficient and the rate of heat transfer have been found in this investigation. An increase in the values of Joule heating parameter $J$ leads to an increase in both the velocity and temperature profiles. The local skin friction coefficient $f''(\xi,0)$ increases at different positions of $\xi$ and also the local rate of heat transfer $\theta'(\xi,0)$ increases at different positions of $\xi$ for $Pr$ =0.72, $\varepsilon$ = 1.00 and $N$ = 0.60.
- Increasing values of Prandtl's number $Pr$ decrease the velocity profiles. The temperature profiles, the local skin friction coefficient $f''(\xi,0)$ and also the local rate of heat transfer $\theta'(\xi,0)$ increase with the increase in Prandtl's number $Pr$ when $J$ = 0.60, $\varepsilon$ = 1.00 and $N$ = 0.50.

## Competing Interests

Authors have declared that no competing interests exist.

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